Inclusive breakup of three-body projectiles: A unified four-body framework for pair-detection and single-particle observables

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Inclusive breakup measurements detect only part of a projectile while the remaining fragments interact with the target. For a projectile with three constituents, these measurements can probe correlations that are lost when the projectile is treated as two preformed clusters. I derive cross-section sum rules within the distorted-wave Born approximation for detecting either a pair or a single constituent, retaining the full three-body projectile wave function. For pair detection, the detected state is projected from this wave function. Comparing the resulting reaction source with its factorized limit tests, at the amplitude level, what the preformed-cluster approximation omits. For single-particle detection, both unobserved constituents can interact with the target, and their absorption includes a contribution mediated by target excitation that is absent from an independent-fragment description. The optical reduction recovers the Carlson-Frederico-Hussein absorption kernel [Phys. Lett. B **767**, 53 (2017)]; explicit coupling of the detected particle to target excitation adds further contributions. The $^6\mathrm{Li} = \alpha + n + p$ example shows how neutron-proton states beyond the bound deuteron enter the reaction source. I give a route for calculating the single-particle spectrum using inelastic target transition form factors and three-body propagation.

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I. INTRODUCTION

A deuteron can break up in a collision with a nucleus, leaving an outgoing proton while the neutron scatters from the target, excites it, or is captured. If only the proton is measured, its spectrum contains contributions from all these neutron-target final states. This is an inclusive breakup measurement [1]. It gives access to reactions of the unobserved neutron through the detected proton. The theoretical task is to calculate this spectrum by summing over the unobserved states, while retaining the interactions that determine their contributions.

Early closed-form treatments used unitarity and surface approximations [2–5]. For a two-body projectile $a = b + x$, the unobserved $x + A$ channels can be summed by completeness, which expresses the inclusive nonelastic breakup (NEB) cross section through the imaginary part of an optical Green's function acting on a breakup source. This is the basis of the Ichimura-Austern-Vincent (IAV) sum rule and its prior form, the Udagawa-Tamura (UT) and Hussein-McVoy decomposition [6–11]. The resulting theory is now used routinely for (d, p) and $(^6\mathrm{Li}, \alpha)$ reactions [12–20]. The prior form is convenient for numerical calculations because the post-form transition integral lacks a natural cutoff at large distances and converges slowly [6,21]. Eikonal and Faddeev formulations provide complementary treatments of the same two-body projectile problem [22,23].

Many weakly bound nuclei require three active projectile constituents. Examples include the Borromean halos $^6\mathrm{He} = \alpha + n + n$ and $^{11}\mathrm{Li} = {}^9\mathrm{Li} + n + n$, the $\alpha + n + p$ structure of ^{6}Li when deuteron breakup matters, and $^9\mathrm{Be} = \alpha + \alpha + n$ [16,24–28]. Reducing these projectiles to two clusters hides the effect of their three-body structure and offers no internal test of that reduction.

For $a = i + j + k$, two different inclusive measurements must be distinguished. Detecting a correlated pair $b = (ij)$ leaves a two-body $k + A$ system unresolved, $a + A \to b + (k + A)^*$. Detecting only i leaves the three-body $jk + A$ system unresolved, $a + A \to i + (jk + A)^*$. The first channel is close to the usual IAV problem once its source has been constructed; the second requires a three-body propagator. For ^{6}Li, inclusive deuteron detection leaves $\alpha + A$, whereas proton detection leaves $n + \alpha + A$.

Carlson, Frederico, and Hussein (CFH) treated the single-particle channel by reducing the target-ground-state part of the $jk + A$ propagator to a three-body optical form with absorptive terms $W_j + W_k + W_{3B}$ [29–31]. The cross term W_{3B} has no counterpart for a two-body projectile. Existing applications nevertheless use two-body cluster models [32]. The complementary pair-detection channel has not been formulated as an IAV sum rule with the detected pair projected from a full three-body projectile. Treating the pair as preformed, as in reduced descriptions of $^{197}\mathrm{Au}(^9\mathrm{Be}, {}^8\mathrm{Be})$ [27], removes precisely the pair mixing that such a measurement can test.

I derive both channels from one four-body Hamiltonian, retaining the full three-body projectile wave function. For

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pair detection, the source contains the interactions of the pair with both the third particle and the target. Projection onto the measured pair state gives a coincidence cross section and allows the preformed-cluster approximation to be tested by factorizing the projectile wave function. That factorization recovers the detected-composite formalism of Ref. [33]. For single-particle detection, the target-ground-state optical reduction recovers CFH. Retaining $V_{iA} - U_{iA}$ in the source adds target-excited contributions, whose direct part can be reduced with a diagonal-intermediate-target-state approximation. The interference terms remain in Feshbach form.

The sum rules use the distorted-wave Born approximation (DWBA) for the exit-channel residual interaction. The unresolved propagators are retained before the optical reductions are made. In the two-body-projectile limit, the results can be compared with the exact Faddeev treatment of Ref. [23]; an exact treatment of a three-body projectile would require four-body Yakubovsky equations. The derivations below separate the sum rules from the approximations needed to evaluate them. The ^{6}Li example shows what is lost in a two-cluster description, and the discussion gives a numerical route for each observable and the limits of the target reduction.

II. FOUR-BODY HAMILTONIAN

I consider the inclusive breakup reaction induced by a three-body projectile $a = i + j + k$ on a target nucleus A. The full Hamiltonian for the four-body $a + A$ system is

$$H = H_A(\xi) + h_i + h_j + h_k + K_i + K_j + K_k + V_{ij} + V_{ik} + V_{jk} + V_{iA} + V_{jA} + V_{kA}, \tag{1}$$

where $H_A(\xi)$ is the internal Hamiltonian of the target nucleus with ground state $H_A\Phi_A = E_A\Phi_A$ and internal coordinates ξ; h_i, h_j, and h_k are the internal Hamiltonians of the three projectile constituents when they are composite; K_i, K_j, and K_k are their center-of-mass kinetic energies; V_{ij}, V_{ik}, and V_{jk} are the mutual interactions among the constituents; and V_{iA}, V_{jA}, and V_{kA} are their interactions with the target. For a composite constituent, h_i acts on its internal coordinates ζ_i, and likewise for j and k. The pairwise interactions may depend on these coordinates as well as on the relative positions.

The projectile a is a bound state of its three constituents. To describe its internal structure, I introduce two sets of Jacobi coordinates adapted to the two exit-channel partitions.

For the pair-detection channel, in which the pair $b \equiv (ij)$ is detected and k is unresolved, the natural Jacobi coordinates are

$$\boldsymbol{\zeta} = \mathbf{r}_i - \mathbf{r}_j, \tag{2}$$

$$\mathbf{y} = \mathbf{r}_k - \frac{m_i\mathbf{r}_i + m_j\mathbf{r}_j}{m_i + m_j}, \tag{3}$$

where $\boldsymbol{\zeta}$ is the internal coordinate of the pair and $\mathbf{y}$ is the relative coordinate of k with respect to the pair's center of mass. The kinetic energies transform as $K_i + K_j + K_k = K_a + K_\zeta + K_y$, where K_a is the center-of-mass kinetic energy of a, K_ζ is the kinetic energy conjugate to $\boldsymbol{\zeta}$ with reduced mass $\mu_\zeta = m_i m_j/(m_i + m_j)$, and K_y is the kinetic energy conjugate to $\mathbf{y}$ with reduced mass $\mu_y = m_k(m_i + m_j)/(m_i + m_j + m_k)$. The internal Hamiltonian of the pair is $h_b = h_i + h_j + K_\zeta + V_{ij}$, with eigenstates $h_b|\phi_\alpha\rangle = \varepsilon_\alpha|\phi_\alpha\rangle$. The bound states of b (if any) correspond to discrete values of α, while continuum states are labeled by the relative momentum $\mathbf{q}$ of the pair's constituents. Here and below, the symbol $\sum_\alpha$ is used as a compact notation for the sum over discrete bound states plus the integral over continuum states. These eigenstates form a complete set in b's internal Hilbert space, $\sum_\alpha |\phi_\alpha\rangle\langle\phi_\alpha| = \mathbf{1}_\zeta$.

For the single-particle channel in which particle i is detected and the pair (jk) is unresolved, the natural Jacobi coordinates are

$$\boldsymbol{\zeta}' = \mathbf{r}_j - \mathbf{r}_k, \tag{4}$$

$$\mathbf{y}' = \mathbf{r}_i - \frac{m_j\mathbf{r}_j + m_k\mathbf{r}_k}{m_j + m_k}. \tag{5}$$

Expressing the constituent positions in either Jacobi set gives

$$\begin{pmatrix}\boldsymbol{\zeta}'\\ \mathbf{y}'\end{pmatrix} = \mathcal{J}\begin{pmatrix}\boldsymbol{\zeta}\\ \mathbf{y}\end{pmatrix}, \quad \mathcal{J} = \begin{pmatrix} -\frac{m_i}{m_i+m_j} & -1 \\ \frac{m_j M_a}{(m_i+m_j)(m_j+m_k)} & -\frac{m_k}{m_j+m_k}\end{pmatrix}, \tag{6}$$

where $M_a = m_i + m_j + m_k$. The determinant of the matrix in Eq. (6) is unity, $\det\mathcal{J} = m_i m_k/[(m_i + m_j)(m_j + m_k)] + m_j M_a/[(m_i + m_j)(m_j + m_k)] = 1$, so the transformation preserves the internal volume element. The unscaled coordinates are not related by an orthogonal rotation. For the $^6\mathrm{Li} = \alpha + n + p$ specialization with $m_n \approx m_p$ and $m_i = m_n$, $m_j = m_p$, $m_k = m_\alpha$, Eq. (6) gives $\boldsymbol{\zeta}' = -\frac{1}{2}\boldsymbol{\zeta} - \mathbf{y}$ and $\mathbf{y}' = \frac{M_a}{2(m_p+m_\alpha)}\boldsymbol{\zeta} - \frac{m_\alpha}{m_p+m_\alpha}\mathbf{y}$, so both coordinates mix with coefficients of order unity. The same projectile state has equal normalization in the two sets, $\Phi_a(\boldsymbol{\zeta}, \mathbf{y}) = \Phi_a(\boldsymbol{\zeta}', \mathbf{y}')$, but a product $\Phi_a \approx \phi_0(\boldsymbol{\zeta})f(\mathbf{y})$ generally ceases to factorize after this transformation. A cluster approximation suited to pair detection therefore need not hold for single-particle detection.

The projectile bound-state wave function $\Phi_a(\boldsymbol{\zeta}, \mathbf{y})$, or equivalently $\Phi_a(\boldsymbol{\zeta}', \mathbf{y}')$ after a Jacobi transformation, satisfies

$$(K_\zeta + K_y + V_{ij} + V_{ik} + V_{jk} + h_i + h_j + h_k)\,\Phi_a = -\epsilon_a\,\Phi_a, \tag{7}$$

where $\epsilon_a > 0$ is the projectile binding energy and the internal Hamiltonians h_i, h_j, h_k act on the respective internal coordinates. For structureless constituents, $h_i = h_j = h_k = 0$ and Φ_a depends only on $\boldsymbol{\zeta}$ and $\mathbf{y}$. The total energy is $E = E_a - \epsilon_a + E_A$, where $E_a = \hbar^2 k_a^2/(2\mu_a)$ is the entrance-channel kinetic energy with reduced mass $\mu_a = m_a m_A/(m_a + m_A)$. Throughout this paper I take the target ground-state energy as the energy origin, $E_A = 0$, so that $E = E_a - \epsilon_a$ in all subsequent formulas and the kinematic energies E_b, E_i, $E_{k,\alpha}$, E_{jkA} defined below are referred to this origin. Restoring $E_A \neq 0$ amounts to a uniform additive shift of $\mathcal{H}_{PP}$ in Eq. (49) and of all target-excited energies $\omega_{A'}$ in Eqs. (64)–(68) by the same constant. The entrance-channel distorted wave $\chi_a^{(+)}(\mathbf{R})$ is the solution of $(K_a + U_a - E_a)\chi_a^{(+)} = 0$ with outgoing asymptotic boundary conditions, where $\mathbf{R}$ is the projectile-target relative coordinate and U_a is the entrance-channel optical potential describing elastic $a + A$ scattering. Analogously, the exit-channel distorted waves $\chi_b^{(-)}$ and $\chi_i^{(-)}$ satisfy the corresponding exit-channel optical-model equations with incoming boundary conditions and kinetic energies E_b, E_i, respectively. I retain the finite target mass and use relative

Jacobi coordinates in the overall center-of-mass frame. The natural exit-channel Jacobi coordinates are $\mathbf{r}_b$, the detected-pair center of mass (c.m.) relative to the center of mass of the residual $k+A$ system in Partition A, and $\mathbf{r}_i$, the detected-particle position relative to the center of mass of the residual $jk+A$ system in Partition B. They are conjugate to the exit-channel kinetic operators K_b and K_i and are the arguments of the distorted waves $\chi_b^{(-)}$, $\chi_i^{(-)}$ and of the auxiliary potentials U_b, U_i that generate them. The microscopic interactions V_{iA}, V_{jA}, V_{kA}, by contrast, are local in the fragment-target separations, which differ from the Jacobi coordinates by the recoil of the residual system,

$$\mathbf{r}_{bA} = \mathbf{r}_b + \frac{m_k}{m_k+m_A}\,\mathbf{r}_{kA},$$
$$\mathbf{r}_{iA} = \mathbf{r}_i + \frac{m_j+m_k}{m_j+m_k+m_A}\,\mathbf{r}_{(jk)A}, \tag{8}$$

where $\mathbf{r}_{bA}$ ($\mathbf{r}_{iA}$) is the position of the pair center of mass (of particle i) relative to the target, $\mathbf{r}_{kA} = \mathbf{r}_k - \mathbf{r}_A$ is the unresolved k–A relative coordinate in Partition A, and $\mathbf{r}_{(jk)A}$ is the corresponding (jk)–c.m.–A coordinate in Partition B. The fragment-target and exit Jacobi coordinates coincide only as $m_A \to \infty$. I use Eq. (8) in the source integrals [Eqs. (22), (21), (90), and (101)] and multipole expansions [Eqs. (93), (84)]. The recoil shift adds a dependence on the residual coordinate $\mathbf{r}_{kA}$ or $\mathbf{r}_{(jk)A}$ to the interaction inside the source integral. These are already coordinates of the unresolved system: in Partition A, $\mathbf{r}_{kA}$ is the output coordinate $\mathbf{r}_k$ of $\rho_\alpha^{(A)}(\mathbf{r}_k, \xi)$. Recoil thus introduces no additional integration variable.

The entrance-channel arguments follow from the same exact geometry: in Partition A,

$$\mathbf{y} = \mathbf{r}_{kA} - \mathbf{r}_{bA}, \quad \mathbf{R} = \frac{m_b\mathbf{r}_{bA} + m_k\mathbf{r}_{kA}}{M_a}, \tag{9}$$

with $m_b = m_i + m_j$, and analogously $\mathbf{y}' = \mathbf{r}_{iA} - \mathbf{r}_{(jk)A}$ and $\mathbf{R} = [m_i\mathbf{r}_{iA} + (m_j+m_k)\,\mathbf{r}_{(jk)A}]/M_a$ in Partition B. These linear transformations have unit Jacobian.

For kinetic operators, the symbols K_i, K_j, K_k in Eq. (1) are laboratory-frame single-particle kinetic energies; in any subsystem Hamiltonian they are replaced by reduced-mass kinetic operators conjugate to the appropriate Jacobi coordinates. Specifically, in $H_{\rm exit}^{(A)}$ [Eq. (12)], K_b has reduced mass $\mu_b = m_b m_{kA}/(m_b+m_{kA})$ with $m_b = m_i + m_j$, $m_{kA} = m_k + m_A$; in H_{kA}, the operator K_k has reduced mass $\mu_{kA} = m_k m_A/(m_k+m_A)$; in $H_{\rm exit}^{(B)}$ [Eq. (40)], K_i has reduced mass $\mu_i = m_i m_{jkA}/(m_i+m_{jkA})$ with $m_{jkA} = m_j + m_k + m_A$; and in H_{jkA} [Eq. (41)] $K_j + K_k$ stand for the two reduced-mass kinetic operators conjugate to the jk internal and (jk)–A relative coordinates.

For later use, I define the composite interactions between the pair $b = (ij)$ and the remaining particle k and target A:

$$V_{bk} = V_{ik} + V_{jk}, \tag{10}$$
$$V_{bA} = V_{iA} + V_{jA}. \tag{11}$$

These interactions depend on the pair's internal coordinate $\boldsymbol{\zeta}$ because the positions of i and j relative to the target or to k depend on their separation within the pair.

III. PAIR-DETECTION INCLUSIVE BREAKUP

I first consider the pair-detection channel $a + A \to b + (k+A)^*$, in which the correlated pair $b = (ij)$ is detected in a definite internal state ϕ_α while the remaining particle k and target A are unresolved. The pair is selected by projecting the full projectile wave function onto ϕ_α; it need not be a preformed entrance-channel cluster. Summing over the unobserved $k+A$ states then follows the usual IAV construction.

The exit-channel Hamiltonian for this partition is

$$H_{\rm exit}^{(A)} = h_b + K_b + U_b + H_{kA}, \tag{12}$$

where $h_b = h_i + h_j + K_\zeta + V_{ij}$ is the internal Hamiltonian of the pair, K_b is the center-of-mass kinetic energy of b, U_b is an auxiliary optical potential that generates the distorted wave $\chi_b^{(-)}$ for the pair, and $H_{kA} = H_A + h_k + K_k + V_{kA}$ is the full $k+A$ Hamiltonian. This exit-channel Hamiltonian is separable in b's degrees of freedom and the $k+A$ system, so the exit-channel eigenstates factorize as $|\chi_b^{(-)}(\mathbf{k}_b)\rangle\,|\phi_\alpha\rangle\,|\Psi_{kA}^c\rangle$. The post-form residual interaction is

$$V_{\rm post}^{(A)} = H - H_{\rm exit}^{(A)} = V_{bk} + V_{bA} - U_b, \tag{13}$$

Both V_{bk} and V_{bA} retain their dependence on $\boldsymbol{\zeta}$ through the positions of i and j.

The DWBA transition amplitude for detecting b in internal state ϕ_α while the $k+A$ system is in eigenstate c is

$$T_{\alpha,c}^{(A)} = \langle \chi_b^{(-)}\phi_\alpha\,\Psi_{kA}^c|\,V_{\rm post}^{(A)}\,|\chi_a^{(+)}\Phi_a\Phi_A\rangle. \tag{14}$$

The state-resolved doubly differential inclusive cross section, summed over all unobserved final states of the $k+A$ system, is

$$\frac{d^2\sigma_\alpha^{(A)}}{dE_b\,d\Omega_b} = \frac{(2\pi)^4}{v_a}\sum_c \left|T_{\alpha,c}^{(A)}\right|^2 \delta(E_{k,\alpha} - E^c), \tag{15}$$

where v_a is the a–A relative velocity and $E_{k,\alpha} = E$–E_b–ε_α is the energy available to the $k+A$ system when the pair is in internal state ϕ_α with center-of-mass kinetic energy E_b. The factor $(2\pi)^4/v_a$ corresponds to the standard IAV normalization in which momentum eigenstates are normalized as $\langle\mathbf{k}|\mathbf{k}'\rangle = \delta(\mathbf{k}-\mathbf{k}')$ (plane-wave δ normalization) and the detected-fragment phase-space factor is included in the chosen differential T-matrix convention. Throughout, the entrance and exit distorted waves $\chi_a^{(+)}(\mathbf{R})$, $\chi_b^{(-)}(\mathbf{r}_b)$, $\chi_i^{(-)}(\mathbf{r}_i)$ are momentum-normalized in the same convention, with asymptotic plane-wave amplitudes of unit strength; the corresponding $\delta(\mathbf{k}-\mathbf{k}')$ normalization on the asymptotic momentum carries through to the on-shell T-matrix and to the spectral identity for the unresolved-subsystem resolvent. Using a different scattering-state normalization would multiply all sum-rule expressions below by the same kinematic factor and would not alter any source or kernel identity.

I define the source function by projecting onto the pair's internal state and the distorted wave,

$$\left|\rho_\alpha^{(A)}\right\rangle = \left\langle\phi_\alpha\,\chi_b^{(-)}\right|\,V_{\rm post}^{(A)}\left|\chi_a^{(+)}\Phi_a\Phi_A\right\rangle, \tag{16}$$

which is a state in the $k+A$ Hilbert space depending on $\mathbf{r}_k$ and ξ; here and below, $\mathbf{r}_k$ denotes the k–A relative coordinate $\mathbf{r}_{kA}$ of Eq. (8), with the subscript A dropped on this variable

for brevity. Substitution into Eq. (15) gives

$$\frac{d^2\sigma_\alpha^{(A)}}{dE_b\,d\Omega_b} = \frac{(2\pi)^4}{v_a}\sum_c \left\langle \rho_\alpha^{(A)}\middle|\Psi_{kA}^c\right\rangle\left\langle\Psi_{kA}^c\middle|\rho_\alpha^{(A)}\right\rangle \delta(E_{k,\alpha}-E^c), \tag{17}$$

so that the summation over unobserved final states $\{|\Psi_{kA}^c\rangle\}$ is an on-shell projection. Using the spectral representation of the resolvent $G_{kA}^{\rm full}(z)=\sum_c |\Psi_{kA}^c\rangle(z-E^c)^{-1}\langle\Psi_{kA}^c|$ and the Sokhotski-Plemelj identity $(E-H_{kA}+i0^+)^{-1}-(E-H_{kA}-i0^+)^{-1}=-2\pi i\delta(E-H_{kA})$, one has $\sum_c|\Psi_{kA}^c\rangle\delta(E-E^c)\langle\Psi_{kA}^c|=-\pi^{-1}\mathrm{Im}\,G_{kA}^{\rm full}(E+i0^+)$, which converts the on-shell sum into the imaginary part of the resolvent and yields the master sum rule

$$\frac{d^2\sigma_\alpha^{(A)}}{dE_b\,d\Omega_b} = -\frac{(2\pi)^4}{\pi\,v_a}\,\mathrm{Im}\left\langle\rho_\alpha^{(A)}\middle|\,G_{kA}^{\rm full}\middle|\rho_\alpha^{(A)}\right\rangle. \tag{18}$$

where $G_{kA}^{\rm full}=(E_{k,\alpha}^+-H_{kA})^{-1}$ is the full $k+A$ resolvent. This sum rule is exact once the DWBA transition amplitude [Eq. (14)] is adopted. The resolvent $G_{kA}^{\rm full}$ describes two-body relative motion of k and A on top of the internal target dynamics carried by H_A. The fully two-body optical propagator G_k used in the practical formulas below arises only after Feshbach projection onto the target ground state. All the physics of the three-body projectile and the pair's internal structure enters through the source $\rho_\alpha^{(A)}$.

For later comparison with optical-model calculations, I introduce a reference interaction U_{bA} that is local in the exact pair-target separation $\mathbf{r}_{bA}$ [Eq. (8)] and is diagonal in the target ground-state space. It need not be identical to the distorting potential U_b, which is a function of the Jacobi coordinate $\mathbf{r}_b$, although choosing the two with the same functional form is often convenient in practical calculations. Because $V_{\rm post}^{(A)}=(V_{bk}+U_{bA}-U_b)+(V_{bA}-U_{bA})$, the source decomposes into a reference-channel part and an explicit pair-target coupling part, $|\rho_\alpha^{(A)}\rangle=|\rho_\alpha^{(A,\rm ref)}\rangle+|\rho_\alpha^{(A,\rm coup)}\rangle$, where

$$\left|\rho_\alpha^{(A,\rm ref)}\right\rangle = \langle\phi_\alpha\,\chi_b^{(-)}|\,(V_{bk}+U_{bA}-U_b)\,|\chi_a^{(+)}\Phi_a\Phi_A\rangle, \tag{19}$$

$$\left|\rho_\alpha^{(A,\rm coup)}\right\rangle = \langle\phi_\alpha\,\chi_b^{(-)}|\,(V_{bA}-U_{bA})\,|\chi_a^{(+)}\Phi_a\Phi_A\rangle. \tag{20}$$

None of the terms in $V_{bk}+U_{bA}-U_b$ excites the target. The interaction $V_{bk}=V_{ik}+V_{jk}$ depends on $\boldsymbol{\zeta}$ and couples different pair states. The local term $U_{bA}-U_b$ leaves the target unchanged. If these potentials have the same functional form, their difference comes only from the recoil shift between $\mathbf{r}_{bA}$ and $\mathbf{r}_b$ [Eq. (8)] and vanishes as $m_A\to\infty$. The remaining interaction $V_{bA}-U_{bA}$ depends on both $\boldsymbol{\zeta}$ and the target coordinates ξ. Its source is

$$\rho_\alpha^{(A,\rm coup)}(\mathbf{r}_k,\xi) = \int d\mathbf{r}_b\,d\boldsymbol{\zeta}\;\chi_b^{(-)*}\,\phi_\alpha^*[V_{bA}(\mathbf{r}_{bA},\boldsymbol{\zeta},\xi) - U_{bA}(\mathbf{r}_{bA})]\chi_a^{(+)}\,\Phi_a\,\Phi_A(\xi), \tag{21}$$

where the integration runs over the Jacobi coordinate $\mathbf{r}_b$ and the pair internal coordinate $\boldsymbol{\zeta}$, while V_{bA} and U_{bA} are evaluated at the exact pair-target separation $\mathbf{r}_{bA}=\mathbf{r}_b+\frac{m_k}{m_k+m_A}\mathbf{r}_k$ via the recoil relation [Eq. (8)], with $\mathbf{r}_k$ the output variable of the source. The arguments of $\chi_a^{(+)}$ and Φ_a are likewise the exact combinations $\mathbf{R}$ and $\mathbf{y}$ of Eq. (9). For the explicit pair-target interaction, $V_{bA}=V_{iA}+V_{jA}$ with

$$V_{iA}=V_{iA}\left(\mathbf{r}_{bA}+\frac{m_j}{m_i+m_j}\boldsymbol{\zeta},\;\xi\right),$$
$$V_{jA}=V_{jA}\left(\mathbf{r}_{bA}-\frac{m_i}{m_i+m_j}\boldsymbol{\zeta},\;\xi\right), \tag{22}$$

The arguments are the constituent-target separations, including the recoil shift in $\mathbf{r}_{bA}$ from Eq. (8).

The cross section separates into three terms,

$$\frac{d^2\sigma_\alpha^{(A)}}{dE_b\,d\Omega_b} = \sigma_\alpha^{(\rm ref)}+\sigma_\alpha^{(\rm int)}+\sigma_\alpha^{(\rm coup)}, \tag{23}$$

Equation (23) follows by expanding the quadratic form in the two source terms; no approximation is made. To evaluate these terms, I split the sources with the target-ground-state projector $P=|\Phi_A\rangle\langle\Phi_A|$ and its complement $Q=\mathbf{1}-P$, both acting in target Hilbert space only. Because the reference-channel operator $V_{bk}+U_{bA}-U_b$ contains no explicit target-excitation piece, the reference source lives entirely in P space, $P|\rho_\alpha^{(A,\rm ref)}\rangle=|\rho_\alpha^{(A,\rm ref)}\rangle$. The explicit coupling operator $V_{bA}-U_{bA}$, on the other hand, has both a component within P and transitions from P to Q, so

$$\left|\rho_\alpha^{(A,\rm coup)}\right\rangle = \left|\rho_\alpha^{(A,\rm coup),P}\right\rangle + \left|\rho_\alpha^{(A,\rm coup),Q}\right\rangle, \tag{24}$$

with the P and Q components defined by acting with P or Q on the target side of $V_{bA}\,\Phi_A$ before contracting with $\langle\phi_\alpha\,\chi_b^{(-)}|$,

$$\left|\rho_\alpha^{(A,\rm coup),P}\right\rangle = |\Phi_A\rangle\,\langle\phi_\alpha\,\chi_b^{(-)}|(\langle\Phi_A|V_{bA}|\Phi_A\rangle-U_{bA}) \times|\chi_a^{(+)}\,\Phi_a\rangle,$$
$$\left|\rho_\alpha^{(A,\rm coup),Q}\right\rangle = \langle\phi_\alpha\,\chi_b^{(-)}|\,[Q\,V_{bA}\,|\Phi_A\rangle]\,|\chi_a^{(+)}\,\Phi_a\rangle. \tag{25}$$

The P-diagonal piece vanishes identically when U_{bA} is chosen to coincide with the ground-state folding potential $\langle\Phi_A|V_{bA}|\Phi_A\rangle$; for any other choice it is the difference between the folding and reference potentials acting in the source, as in Ref. [33]. The Q component describes target excitation by V_{bA}, which is absent from ordinary closure-based IAV.

To decompose $G_{kA}^{\rm full}$, first define the resolvent restricted to the excited-target space,

$$g_Q^{(A)}=(E_{k,\alpha}^+-QH_{kA}Q)^{-1}. \tag{26}$$

Then

$$\begin{aligned} G_{PP}^{\rm exact} &= PG_{kA}^{\rm full}P,\\ G_{PQ} &= G_{PP}^{\rm exact}\,PV_{kA}Q\,g_Q^{(A)},\\ G_{QP} &= g_Q^{(A)}\,QV_{kA}P\,G_{PP}^{\rm exact},\\ G_{QQ}^{\rm full} &= g_Q^{(A)}+g_Q^{(A)}QV_{kA}P\,G_{PP}^{\rm exact}\,PV_{kA}Q\,g_Q^{(A)}. \end{aligned} \tag{27}$$

The full Q-Q block contains the direct propagation in the target-excited space plus $Q\to P\to Q$ rescattering through the target-ground-state block. I abbreviate the prefactor and sources as

$$\begin{aligned} C_a &= \frac{(2\pi)^4}{\pi\,v_a},\\ |\rho_r\rangle &= \left|\rho_\alpha^{(A,\rm ref)}\right\rangle,\\ |\rho_P\rangle &= \left|\rho_\alpha^{(A,\rm coup),P}\right\rangle,\\ |\rho_Q\rangle &= \left|\rho_\alpha^{(A,\rm coup),Q}\right\rangle. \end{aligned}$$

In the following expressions I write the P-sector block as G_k when it is represented by the usual elastic optical propagator, with the dynamic polarization from eliminated target excitations absorbed into U_k. This replacement is the same optical reduction used in ordinary IAV; it is not part of the unreduced DWBA identity in Eq. (18). The off-diagonal blocks G_{PQ}, G_{QP} and the full block $G^{\rm full}_{QQ}$ are kept as Feshbach-block notation unless an additional reduction is stated:

$$\sigma_\alpha^{(\rm ref)} = -C_a\,\mathrm{Im}\,\langle\rho_r|G_k|\rho_r\rangle, \tag{28}$$

$$\sigma_\alpha^{(\rm int)} = -C_a\,\mathrm{Im}\,[\langle\rho_r|G_k|\rho_P\rangle + \langle\rho_P|G_k|\rho_r\rangle + \langle\rho_r|G_{PQ}|\rho_Q\rangle + \langle\rho_Q|G_{QP}|\rho_r\rangle], \tag{29}$$

$$\sigma_\alpha^{(\rm coup)} = -C_a\,\mathrm{Im}\,\big[\langle\rho_P|G_k|\rho_P\rangle + \langle\rho_P|G_{PQ}|\rho_Q\rangle + \langle\rho_Q|G_{QP}|\rho_P\rangle + \langle\rho_Q|G^{\rm full}_{QQ}|\rho_Q\rangle\big]. \tag{30}$$

Equation (30) contains propagation in the target ground state, interference between ground and excited target states, and propagation between excited target states. Changing U_{bA} redistributes strength between ρ_r and ρ_P, so the separated contributions are not observable. Their sum is fixed by $\rho_\alpha^{(A)}$. The last term cannot be absorbed into a choice of the elastic reference interaction U_{bA}. The interference terms have been kept as explicit G_{PQ} and G_{QP} matrix elements because the outgoing resolvent is not a Hermitian operator; only after a further reduction to an absorptive kernel can they be recast as real interference terms. For the usual elastic-breakup (EBU) and NEB separation, I first restrict the source to the target ground state. The excited-target piece $\rho_\alpha^{(A,\rm coup),Q}$ of Eq. (25) is dropped, leaving the single-channel total source $\tilde\rho_\alpha^{(A,\rm tot)} = \tilde\rho_\alpha^{(A,\rm ref)} + \tilde\rho_\alpha^{(A,\rm coup),P}$ entirely in the target-ground-state P sector. I then apply the two-potential (or Kawai) identity [6,34], which represents the imaginary part of the full $k+A$ resolvent in the optical-model approximation as

$$-\pi^{-1}\mathrm{Im}\,G^{\rm full}_{kA} \approx |\chi_k^{(+)}\rangle\langle\chi_k^{(+)}|_{\rm on\text{-}shell} - \pi^{-1}\,G_k^\dagger\,W_k\,G_k, \tag{31}$$

where $G_k = (E^+_{k,\alpha} - K_k - U_k)^{-1}$ is the elastic-channel optical Green's function for the $k+A$ system, $U_k = V_k + iW_k$ with $W_k \leqslant 0$, and $|\chi_k^{(+)}\rangle$ is the outgoing optical distorted wave at energy $E_{k,\alpha}$ (with $E_A = 0$ from Sec. II; restoring $E_A \neq 0$ adds $-E_A$ to the denominator). The first term is the elastic-channel on-shell density of $G^{\rm full}_{kA}$, identified at the optical level with $|\chi_k^{(+)}\rangle\langle\chi_k^{(+)}|$, and is not extractable from $\mathrm{Im}\,G_k$ alone (the optical G_k has no real-axis pole); the second is the inelastic spectral density of $G^{\rm full}_{kA}$ approximated by the absorptive part of G_k, which follows from the algebraic identity $G_k - G_k^\dagger = 2i\,G_k^\dagger W_k G_k$ and represents the nonelastic flux loss in the optical sense. Both pieces contribute to the master formula [Eq. (18)] when the source is restricted to its P sector, yielding the decomposition into NEB and EBU

$$\sigma_\alpha^{(A,\rm NEB)} = -\frac{(2\pi)^4}{\pi\,v_a}\big\langle\tilde\rho_\alpha^{(A,\rm tot)}\big|G_k^\dagger W_k G_k\big|\tilde\rho_\alpha^{(A,\rm tot)}\big\rangle, \tag{32}$$

$$\sigma_\alpha^{(A,\rm EBU)} = \frac{(2\pi)^4}{v_a}\big|\big\langle\chi_k^{(+)}\big|\tilde\rho_\alpha^{(A,\rm tot)}\big\rangle\big|^2, \tag{33}$$

$$\left[\frac{d^2\sigma_\alpha^{(A)}}{dE_b\,d\Omega_b}\right]_{\rm 1\text{-}ch} = \sigma_\alpha^{(A,\rm NEB)} + \sigma_\alpha^{(A,\rm EBU)}. \tag{34}$$

The subscript "1-ch" denotes the restriction of the source to P. Equation (34) omits all matrix elements containing ρ_Q: the PQ and QP interferences in Eqs. (29) and (30) and $\langle\rho_Q|G^{\rm full}_{QQ}|\rho_Q\rangle$. These terms remain in the unreduced sum rule.

Equation (32) has the IAV NEB form, with G_k in place of G_x and a pair-projected source. In the EBU term [Eq. (33)], k scatters elastically from the target and the pair exits in state ϕ_α. The two contributions are separate reaction channels within this reduction. For a detected deuteron, $\alpha = 0$ and $\phi_\alpha = \phi_d$, the cluster limit recovers the usual DWBA elastic-breakup amplitude. Both NEB and EBU sample the projectile structure through the source; the propagator remains that of $k+A$.

Choosing $\alpha = 0$ (the pair's ground state, if bound) gives the cross section for detecting an intact pair, while $\alpha = (\mathbf{q})$ (a continuum state with relative momentum $\mathbf{q}$) gives the inclusive coincidence cross section for detecting the pair's breakup products at specified relative energy and angle. The total inclusive cross section, summed over all internal states of the pair, is

$$\frac{d^2\sigma^{(A)}}{dE_b\,d\Omega_b} = \sum_\alpha \frac{d^2\sigma_\alpha^{(A)}}{dE_b\,d\Omega_b}, \tag{35}$$

where the shorthand $\sum_\alpha$ stands for a discrete sum over bound pair states plus a continuum integral. For the discrete part, each $d^2\sigma_\alpha^{(A)}/(dE_b\,d\Omega_b)$ is added directly; the continuum part is added either as a phase-space integral over the density $d^5\sigma/(dE_b\,d\Omega_b\,d^3\mathbf{q})$ in the continuum normalization, or as a sum over discrete-bin cross sections in the bin normalization, using the conventions specified below. Only in the additional approximation that the α dependence of the unresolved propagator is weak over the populated pair states, so that $G^{\rm full}_{k,\alpha} = (E - E_b - \varepsilon_\alpha - H_{kA})^{-1}$ may be replaced by a common reduced propagator $G_k(\bar E_k)$ at some representative energy $\bar E_k = E - E_b - \bar\varepsilon$, with $\bar\varepsilon$ a representative pair-state energy chosen within the populated α-window (e.g., a centroid value), does closure give

$$\frac{d^2\sigma^{(A)}}{dE_b\,d\Omega_b} \approx -\frac{(2\pi)^4}{\pi\,v_a}\,\mathrm{Im}\,\langle\mathcal{S}^{(A)}|\,\mathbf{1}_\zeta \otimes G_k(\bar E_k)\,|\mathcal{S}^{(A)}\rangle, \tag{36}$$

where $\mathcal{S}^{(A)}(\mathbf{r}_k, \boldsymbol{\zeta}) = \langle\chi_b^{(-)}|V^{(A)}_{\rm post}\,\chi_a^{(+)}\Phi_a\rangle_{\mathbf{r}_b}$ is the unprojected source and $\mathbf{1}_\zeta = \sum_\alpha |\phi_\alpha\rangle\langle\phi_\alpha|$ is the pair-Hilbert completeness relation. The replacement $G_{k,\alpha} \to G_k(\bar E_k)$ is the only step beyond exactness in Eq. (36). The operator-level error has the formal form

$$G_{k,\alpha} - G_k(\bar E_k) = G_{k,\alpha}\,(\varepsilon_\alpha - \bar\varepsilon)\,G_k(\bar E_k), \tag{37}$$

which is a resolvent identity, not a uniform norm bound: both $G_{k,\alpha}$ and $G_k(\bar E_k)$ are unbounded operators on the continuous spectrum, so a strict $\|\cdot\|_{\rm op}$ estimate would require restricting to states sufficiently close to the on-shell pole, where the absorptive width $|W_k|$ regulates the resolvent. The schematic dimensional scale relevant for that on-shell region is $\Delta\varepsilon_{\rm pair}/|W_k|$, where $\Delta\varepsilon_{\rm pair} \equiv \sup_{\alpha\in\rm populated}|\varepsilon_\alpha - \bar\varepsilon|$ is the energy spread of the populated pair states relative to the chosen $\bar\varepsilon$, and $|W_k|$ stands for the magnitude of the on-shell absorptive optical potential of the kA system, $\langle\chi_k^{(+)}||W_k|\,|\chi_k^{(+)}\rangle$ (treated as a positive scalar scale here). The corresponding on-shell width $\Gamma_{kA} \equiv -2\langle\chi_k^{(+)}|W_k|\chi_k^{(+)}\rangle = 2|W_k|$ (in the same on-shell

sense) provides the natural absorptive scale. Equation (36) is therefore reliable when $\Delta\varepsilon_{\rm pair} \ll \Gamma_{kA}$, trivial when only a single bound pair state is retained, and degrades once the populated continuum window approaches the kA optical imaginary scale, in agreement with the qualitative discussion of Ref. [33]. The state-resolved sum rule [Eqs. (34) and (35)] bypasses this replacement entirely.

For the coincidence observable, the pair's internal state label α is the relative momentum $\mathbf{q}$ of the constituents. Two normalization conventions are used in practice. The continuum-normalized pair state $|\phi_{\mathbf{q}}\rangle$ satisfies $\langle\phi_{\mathbf{q}}|\phi_{\mathbf{q}'}\rangle_\zeta = \delta^3(\mathbf{q}-\mathbf{q}')$, for which the state-resolved cross section $d^2\sigma^{(A)}_{\mathbf{q}}/(dE_b\,d\Omega_b)$ inherits an extra phase-space factor per unit $\mathbf{q}$ and is more appropriately written as a quintuple differential $d^5\sigma/(dE_b\,d\Omega_b\,d^3\mathbf{q})$. The bin state $|\tilde{\phi}_n\rangle = \mathcal{N}_n^{-1}\int_{q_n}^{q_{n+1}} dq\,q^2\int d\Omega_{\mathbf{q}}\,g_n(\hat{\mathbf{q}})|\phi_{\mathbf{q}}\rangle$, with weight g_n and normalization $\mathcal{N}_n^2 = \int_{q_n}^{q_{n+1}} dq\,q^2\int d\Omega_{\mathbf{q}}|g_n|^2$, satisfies $\langle\tilde{\phi}_n|\tilde{\phi}_m\rangle = \delta_{nm}$, so that $d^2\sigma^{(A)}_{\tilde{\phi}_n}/(dE_b\,d\Omega_b)$ retains the dimensions of an ordinary doubly differential cross section and can be summed as in Eq. (35) with discrete α. The bin cross section is the coherent square of a bin amplitude, not a literal $|g_n|^2$-weighted integral of the continuum density. Equivalently,

$$T^{(A)}_{\tilde{\phi}_n} = \mathcal{N}_n^{-1}\int_{q_n}^{q_{n+1}} dq\,q^2\int d\Omega_{\mathbf{q}}\,g_n(\hat{\mathbf{q}})T^{(A)}_{\mathbf{q}}, \tag{38}$$

and $d^2\sigma^{(A)}_{\tilde{\phi}_n}/(dE_b\,d\Omega_b) = (2\pi)^4 v_a^{-1}|T^{(A)}_{\tilde{\phi}_n}|^2$. Only in the narrow-bin limit, where $T^{(A)}_{\mathbf{q}}$ varies slowly across the bin, does this coherent expression reduce to the integrated continuum density (for the usual top-hat bin weight). Bin and continuum formulations carry the same physical content when the bin widths are small compared with the variation scale of the source and the kA resolvent. One may then transform from the pair center-of-mass variables $(E_b, \Omega_b, \mathbf{q})$ to the laboratory variables of the individual particles $(E_i, \Omega_i, E_j, \Omega_j)$ via the appropriate Jacobian to obtain the corresponding pair-resolved inclusive distribution,

$$\frac{d^6\sigma}{dE_i\,d\Omega_i\,dE_j\,d\Omega_j} = J(\mathbf{q}; E_i, \Omega_i, E_j, \Omega_j)\,\frac{d^5\sigma}{dE_b\,d\Omega_b\,d^3\mathbf{q}}, \tag{39}$$

where J is the kinematic Jacobian and the right-hand side is evaluated at the $(E_b, \Omega_b, \mathbf{q})$ values corresponding to the specified laboratory kinematics through momentum and energy conservation. If a discretized bin state is used instead, Eq. (39) is first applied to the continuum density and then integrated over the bin with the weight appearing in Eq. (35); the bin cross section itself should not be read as a density in $d^3\mathbf{q}$. This observable is semi-inclusive: it is exclusive with respect to the measured kinematics of particles i and j, but remains inclusive with respect to all final states of the unresolved $k+A$ subsystem. Once particles i and j are kinematically measured, four-body energy-momentum conservation together with a ground-state target leaves only a discrete set of kinematic configurations for k; under that specialization the present observable reduces to an exclusive three-body breakup measurement of the type $A(a, ij)k$ [25,27]. The inclusive sum in Eq. (39) runs over target internal states and those k kinematics that are not already fixed by the measured $(\mathbf{k}_i, \mathbf{k}_j)$, i.e., target-excitation channels together with compound-nucleus formation in $k+A$.

The practical Jacobian J is the standard kinematic transformation from the pair center-of-mass variables $(E_b, \Omega_b, \mathbf{q})$ to the individual-particle laboratory variables, determined by momentum conservation inside the pair and independent of the dynamical content of the sum rule. Equation (39) is the laboratory-frame representation of the pair-resolved sum rule; it does not supply an exact three-body asymptotic breakup amplitude.

IV. SINGLE-PARTICLE INCLUSIVE BREAKUP

I now consider the single-particle channel $a+A \to i+(jk+A)^*$, in which particle i is detected while the subsystem $j+k+A$ is unresolved. I treat i as structureless below. For composite i, one retains h_i in the exit Hamiltonian and projects onto its internal state, as in Ref. [33].

The exit-channel Hamiltonian for this partition is

$$H^{(B)}_{\rm exit} = h_i + K_i + U_i + H_{jkA}, \tag{40}$$

where U_i is an auxiliary optical potential generating the distorted wave $\chi_i^{(-)}$ for particle i, and

$$H_{jkA} = H_A + h_j + h_k + K_j + K_k + V_{jk} + V_{jA} + V_{kA}, \tag{41}$$

is the full three-body $jk+A$ Hamiltonian. Within this unresolved subsystem I use $\mathbf{r}_j$ and $\mathbf{r}_k$ to denote the exact j–A and k–A relative coordinates, the natural variables of the folded and absorptive kernels below; the (jk)–c.m.–A coordinate of Eq. (8) is then $\mathbf{r}_{(jk)A} = (m_j\mathbf{r}_j + m_k\mathbf{r}_k)/(m_j+m_k)$, and the kinetic operators $K_j + K_k$ in Eq. (41) remain the reduced-mass Jacobi operators specified in Sec. II. The post-form residual interaction is

$$V^{(B)}_{\rm post} = H - H^{(B)}_{\rm exit} = V_{ij} + V_{ik} + V_{iA} - U_i. \tag{42}$$

The residual contains the interactions of the detected particle with j, k, and A.

The DWBA transition amplitude for detecting i while the $jk+A$ system is in eigenstate c is

$$T^{(B)}_c = \langle\chi_i^{(-)}\Psi^c_{jkA}|\,V^{(B)}_{\rm post}\,|\chi^{(+)}_a\Phi_a\Phi_A\rangle, \tag{43}$$

where $H_{jkA}|\Psi^c_{jkA}\rangle = E^c|\Psi^c_{jkA}\rangle$. Defining the source

$$|\rho^{(B)}_i\rangle = \langle\chi_i^{(-)}|\,V^{(B)}_{\rm post}\,|\chi^{(+)}_a\Phi_a\Phi_A\rangle, \tag{44}$$

and using the spectral identity for the $jk+A$ system, the doubly differential inclusive cross section becomes

$$\frac{d^2\sigma^{(B)}}{dE_i\,d\Omega_i} = -\frac{(2\pi)^4}{\pi v_a}\,{\rm Im}\,\langle\rho^{(B)}_i|\,G_{jkA}\,|\rho^{(B)}_i\rangle, \tag{45}$$

where $G_{jkA} = (E^+_{jkA} - H_{jkA})^{-1}$ is the full three-body $jk+A$ resolvent and $E_{jkA} = E - E_i$ is the energy available to the $jk+A$ system.

Equation (45) follows exactly from the adopted DWBA amplitude [Eq. (43)]. At this stage V_{iA} enters only through the source. Eliminating target-excited sectors can also generate effective couplings in the reduced kernel.

Table I compares the two measurements and their post-form interactions.

TABLE I. Comparison of the two exit-channel partitions for inclusive breakup of a three-body projectile $a = i + j + k$.

	Partition A	Partition B
Detected	Pair $b = (ij)$	Particle i
Unresolved	$k + A$	$jk + A$
Propagator	G_{kA} (two-body)	G_{jkA} (three-body)
$V_{\rm post}$	$V_{bk} + V_{bA} - U_b$	$V_{ij} + V_{ik} + V_{iA} - U_i$
Explicit target coupling	$V_{bA} - U_{bA}$	$V_{iA} - U_{iA}$

Introducing a reference interaction U_{iA}, analogous to U_{bA} in Partition A and likewise local in the exact i–A separation $\mathbf{r}_{iA}$ [Eq. (8)], the source decomposes as $|\rho_i^{(B)}\rangle = |\rho_i^{(B,\mathrm{ref})}\rangle + |\rho_i^{(B,\mathrm{coup})}\rangle$. In the post form [Eq. (42)], V_{iA} enters the source, while V_{jA} and V_{kA} enter H_{jkA} and its propagator. In the prior form, the residual $V_{\rm prior} = V_{iA} + V_{jA} + V_{kA} - U_a$ [Eq. (71)] contains all three fragment-target couplings: only the detected-particle coupling $V_{iA} - U_{iA}$ is assigned to the source, whereas V_{jA}, V_{kA} remain inside the unresolved-system Hamiltonian H_{jkA} and feed the Feshbach kernel of G_{jkA} through $\Delta\mathcal{H}_{PP}$ [Eq. (52)] and the associated off-diagonal and Q-space blocks. Treating $QV_{jA}P$ or $QV_{kA}P$ simultaneously as prior-form source corrections and as Feshbach kernel couplings would double-count the same target-excitation amplitude. The prior-form reduction in Sec. V uses this same assignment.

The reference-channel source is

$$\left|\rho_i^{(B,\mathrm{ref})}\right\rangle = \langle\chi_i^{(-)}|\,(V_{ij} + V_{ik} + U_{iA} - U_i)\,|\chi_a^{(+)}\Phi_a\Phi_A\rangle, \quad (46)$$

built from interactions that do not excite the target, and the explicit target-coupling source

$$\left|\rho_i^{(B,\mathrm{coup})}\right\rangle = \langle\chi_i^{(-)}|\,(V_{iA} - U_{iA})\,|\chi_a^{(+)}\Phi_a\Phi_A\rangle, \quad (47)$$

is the part of the detected-particle–target interaction that is not represented by the elastic reference interaction. If i is structureless, V_{iA} depends only on the exact i–A separation $\mathbf{r}_{iA}$ [Eq. (8)] and ξ, and V_{iA}–U_{iA} represents the difference between the true i–A interaction and the elastic optical model. If i is composite, this operator acquires the same $\boldsymbol{\zeta}_i$ dependence discussed in Ref. [33].

The cross section again separates into reference, interference, and explicit-coupling terms. When V_{iA} is replaced by its target-elastic reference U_{iA}, the explicit coupling source vanishes, the source reduces to $|\rho_i^{(B,\mathrm{ref})}\rangle$, and the result becomes

$$\frac{d^2\sigma^{(B,\mathrm{ref})}}{dE_i\,d\Omega_i} = -\frac{(2\pi)^4}{\pi\,v_a}\,\mathrm{Im}\left\langle\rho_i^{(B,\mathrm{ref})}\right|G_{jkA}\left|\rho_i^{(B,\mathrm{ref})}\right\rangle, \quad (48)$$

which is the DWBA sum-rule starting point of the CFH formalism [29]. I reproduce Eqs. (19)–(23) of Ref. [29] in the present notation because the explicit-coupling terms require the same P-Q decomposition of G_{jkA}. The resulting absorptive kernel is the CFH result, included here for self-containment.

Apply $P = |\Phi_A\rangle\langle\Phi_A|$, $Q = \mathbf{1} - P$ to $G_{jkA} = (E^+_{jkA} - H_{jkA})^{-1}$. With the global $E_A = 0$ convention adopted in Sec. II, $E_{jkA} = E - E_i$ is the energy available to the jk relative motion in the target–ground-state sector. The Feshbach projection of the full target–ground-state block, denoted $G^{\rm exact}_{PP} \equiv P\,G_{jkA}\,P$, satisfies

$$\left(G^{\rm exact}_{PP}\right)^{-1} = E^+_{jkA} - \mathcal{H}_{PP} - \Delta\mathcal{H}_{PP}, \quad (49)$$

with

$$\mathcal{H}_{PP} = h_j + h_k + K_j + K_k + V_{jk} + U_j^{\rm fold} + U_k^{\rm fold},$$
$$U_j^{\rm fold}(\mathbf{r}_j) = \langle\Phi_A|V_{jA}|\Phi_A\rangle, \quad U_k^{\rm fold}(\mathbf{r}_k) = \langle\Phi_A|V_{kA}|\Phi_A\rangle. \quad (50)$$

Equivalent expressions retaining E_A explicitly amount to adding $-E_A$ to $\mathcal{H}_{PP}$ in Eq. (49) and shifting all target-excited energies $\omega_{A'}$ in Eqs. (64)–(68) by the same constant. The reduced Q-space resolvent entering the Feshbach kernel is

$$g_Q \equiv (E^+_{jkA} - QH_{jkA}Q)^{-1}. \quad (51)$$

The polarization kernel is

$$\Delta\mathcal{H}_{PP} = P(V_{jA} + V_{kA})Q\,g_Q\,Q(V_{jA} + V_{kA})P. \quad (52)$$

The operator g_Q in Eq. (52) still contains both fragments and all target-excited channels; the usual optical-model step replaces its one-fragment subblocks by the phenomenological dynamic-polarization contributions that are already included in elastic $j + A$ and $k + A$ optical potentials. The kernel $\Delta\mathcal{H}_{PP}$ is non-Hermitian because g_Q is the retarded resolvent ($+i0^+$ prescription), so its imaginary part is well defined as $\mathrm{Im}\,\Delta\mathcal{H}_{PP} = -\pi\,P(V_{jA} + V_{kA})Q\,\delta(E_{jkA} - QH_{jkA}Q)\,Q(V_{jA} + V_{kA})P$, a negative semidefinite operator on P space. Expanding the square in Eq. (52) produces three distinct contributions,

$$\Delta\mathcal{H}_{PP} = \delta U_j + \delta U_k + \mathcal{V}_{3B}, \quad (53)$$

with

$$\delta U_j = PV_{jA}Q\,g_Q\,QV_{jA}P,$$
$$\delta U_k = PV_{kA}Q\,g_Q\,QV_{kA}P,$$
$$\mathcal{V}_{3B} = PV_{jA}Q\,g_Q\,QV_{kA}P + PV_{kA}Q\,g_Q\,QV_{jA}P. \quad (54)$$

The cross term $\mathcal{V}_{3B}$ has V_{jA} at one end of the propagator and V_{kA} at the other. Its absorptive part, defined symmetrically as

$$W_{3B} \equiv \mathrm{Im}\,\mathcal{V}_{3B} \equiv \frac{1}{2i}(\mathcal{V}_{3B} - \mathcal{V}_{3B}^\dagger), \quad (55)$$

is the genuine four-body absorption identified in Ref. [29], supported where both j and k simultaneously sit near the surface and the inelastic couplings are strong. With the convention $U = V + iW$, $W \leqslant 0$, one has $\mathrm{Im}\,\delta U_j = (PV_{jA}Q)\,\mathrm{Im}\,g_Q\,(QV_{jA}P) \leqslant 0$ as a quadratic form, and analogously for $\mathrm{Im}\,\delta U_k$; $\mathrm{Im}(\Delta\mathcal{H}_{PP}) = P(V_{jA} + V_{kA})Q\,\mathrm{Im}\,g_Q\,Q(V_{jA} + V_{kA})P \leqslant 0$ is the imaginary part of the full polarization kernel. The cross piece W_{3B} alone is *not* sign-definite; only the microscopic combination $W_j^{\rm micro} + W_k^{\rm micro} + W_{3B}$, identified with $\mathrm{Im}(U_j^{\rm fold} + \delta U_j + U_k^{\rm fold} + \delta U_k + \mathcal{V}_{3B})$, inherits the negative semidefiniteness of the full polarization kernel. The phenomenological U_j, U_k obtained from $j + A$ and $k + A$ elastic data already incorporate dynamic polarization not separable into the diagonal Feshbach pieces of Eq. (54); their imaginary parts W_j, W_k are chosen to be absorptive but need not equal $\mathrm{Im}(U_j^{\rm fold} + \delta U_j)$ etc. The negative semidefiniteness of $W_j + W_k + W_{3B}$ as a sum holds rigorously only at the

microscopic Feshbach level; once U_j and U_k are replaced by phenomenological optical potentials, the strict inheritance of negative semidefiniteness relies on the residual W_{3B} being a small perturbation around phenomenologically absorptive $W_j + W_k$, an assumption common to the CFH optical reduction. A sufficient practical condition is that $|\langle\psi|W_{3B}|\psi\rangle| \ll |\langle\psi|(W_j + W_k)|\psi\rangle|$ pointwise in the quadratic-form sense on states $|\psi\rangle$ in the source-supported subspace, which can be checked *a posteriori* from the microscopic decomposition of the polarization kernel [Eq. (54)] before W_{3B} is treated as an absorptive operator alongside W_j and W_k. Under this standard optical reduction, the parts of g_Q in which only j (or only k) couples to target excitations dress the static folding potentials into phenomenological U_j and U_k, with the other constituent remaining inert in that single-coupling piece, while the cross piece $\mathcal{V}_{3B}$ is retained as a genuine three-body kernel. The CFH-optical effective Hamiltonian on the P sector is therefore

$$\mathcal{H}^{\rm opt}_{PP} \equiv h_j + h_k + K_j + K_k + V_{jk} + U_j + U_k + \mathcal{V}_{3B}, \quad (56)$$

and the corresponding optical propagator is

$$G^{\rm opt}_{PP} \equiv \left(E^+_{jkA} - \mathcal{H}^{\rm opt}_{PP}\right)^{-1}, \quad (57)$$

where the optical Hamiltonian replaces the energy-dependent Feshbach polarization in Eq. (49) by an energy-independent representation. The diagonal terms are absorbed into U_j and U_k, while $\mathcal{V}_{3B}$ is retained. Its imaginary part W_{3B} contributes alongside W_j and W_k.

From this point on, G_{PP} refers to the CFH-optical operator $G^{\rm opt}_{PP}$ unless stated otherwise; the exact Feshbach block is denoted $G^{\rm exact}_{PP}$ where the distinction matters. The two-potential identity applied to $G^{\rm opt}_{PP}$ gives

$$\begin{aligned} -\pi^{-1}\,\mathrm{Im}\, G^{\rm opt}_{PP} \approx{}& |\chi^{(+)}_{jk}\rangle\langle\chi^{(+)}_{jk}|_{\rm on\text{-}shell} \\ & - \pi^{-1}\, G^{\rm opt\dagger}_{PP}[W_j + W_k + W_{3B}]G^{\rm opt}_{PP}, \end{aligned} \quad (58)$$

as in Eq. (31). The two terms follow from $G^{\rm opt}_{PP} - G^{\rm opt\dagger}_{PP} = 2i\, G^{\rm opt\dagger}_{PP}[W_j + W_k + W_{3B}]G^{\rm opt}_{PP}$ together with isolation of the elastic-channel on-shell pole, with $W_j = \mathrm{Im}\, U_j$ and $W_k = \mathrm{Im}\, U_k$ in the phenomenological representation. In the microscopic Feshbach limit, these reduce to $\mathrm{Im}(U^{\rm fold}_j + \delta U_j)$ and $\mathrm{Im}(U^{\rm fold}_k + \delta U_k)$, respectively. Here $|\chi^{(+)}_{jk}\rangle$ is the three-body jk elastic-scattering wave function in the ground-state target field. The first term gives the EBU density. It comes from isolating the elastic-channel pole in $\mathrm{Im}(G_{jkA} - G^{\rm opt}_{PP})$, as in Eq. (31). The second gives the NEB density when $W_j + W_k + W_{3B}$ is absorptive. For phenomenological potentials, this sign condition must be checked as discussed below Eq. (54).

The CFH-reduced NEB cross section therefore reads

$$\begin{aligned} &\frac{d^2\sigma^{(B,\rm NEB)}}{dE_i\, d\Omega_i} \\ &\quad = -\frac{(2\pi)^4}{\pi\, v_a}\langle\hat\rho_{jk}|G^{\rm opt\dagger}_{PP}[W_j + W_k + W_{3B}]G^{\rm opt}_{PP}|\hat\rho_{jk}\rangle, \end{aligned} \quad (59)$$

where $\hat\rho_{jk} = \langle\Phi_A|\rho^{(B,\rm ref)}_i\rangle$ is the reference-channel source after projection onto the target ground state. Equations (49)–(59) are the CFH reference-channel absorptive kernel in the present notation. Retaining instead the on-shell elastic projector of Eq. (58) in the master sum rule gives the EBU counterpart

$$\frac{d^2\sigma^{(B,\rm EBU)}}{dE_i\, d\Omega_i} = \frac{(2\pi)^4}{v_a}|\langle\chi^{(+)}_{jk}|\hat\rho_{jk}\rangle|^2. \quad (60)$$

Computing $|\chi^{(+)}_{jk}\rangle$ requires three-body scattering methods, such as those of Ref. [23]. Section VIII discusses how to calculate this state and the action of $G^{\rm opt}_{PP}$ on a source.

To include $V_{iA} - U_{iA}$, I project both sources onto $P = |\Phi_A\rangle\langle\Phi_A|$ and $Q = \mathbf{1} - P$. The reference source lies entirely in P, since $V_{ij} + V_{ik} + U_{iA} - U_i$ does not excite the target. The explicit source has two parts: the difference $\langle\Phi_A|V_{iA}|\Phi_A\rangle - U_{iA}$ within P, and transitions to excited target states through $QV_{iA}P$.

Writing $|\rho^{(B,\rm coup)}_i\rangle = |\rho^{(B,\rm coup),P}_i\rangle + |\rho^{(B,\rm coup),Q}_i\rangle$ with P and Q acting on the target side of $V_{iA}\Phi_A$,

$$\begin{aligned} \left|\rho^{(B,\rm coup),P}_i\right\rangle &= |\Phi_A\rangle\,\langle\chi^{(-)}_i|(\langle\Phi_A|V_{iA}|\Phi_A\rangle - U_{iA}) \\ &\quad \times |\chi^{(+)}_a\Phi_a\rangle, \\ \left|\rho^{(B,\rm coup),Q}_i\right\rangle &= \langle\chi^{(-)}_i|\,[Q\,V_{iA}\,|\Phi_A\rangle]\,|\chi^{(+)}_a\Phi_a\rangle, \end{aligned} \quad (61)$$

which live in the P sector and Q sector of the target Hilbert space, respectively, tensored with the jk relative Hilbert space. The Feshbach decomposition $G_{jkA} = G^{\rm exact}_{PP} + G_{PQ} + G_{QP} + G^{\rm full}_{QQ}$ has blocks

$$\begin{aligned} \left(G^{\rm exact}_{PP}\right)^{-1} &= E_{jkA} - PH_{jkA}P - PH_{jkA}Q\,g_Q\,QH_{jkA}P, \\ G_{PQ} &= G^{\rm exact}_{PP}\,P(V_{jA} + V_{kA})Q\,g_Q, \\ G_{QP} &= g_Q\,Q(V_{jA} + V_{kA})P\,G^{\rm exact}_{PP}, \\ G^{\rm full}_{QQ} &= g_Q + g_Q\,Q(V_{jA} + V_{kA})P\,G^{\rm exact}_{PP} \\ &\quad \times P(V_{jA} + V_{kA})Q\,g_Q, \end{aligned} \quad (62)$$

where g_Q is defined in Eq. (51). These relations are exact before optical reduction. I then replace $G^{\rm exact}_{PP}$ by $G^{\rm opt}_{PP}$ [Eq. (57)], using the energy-independent Hamiltonian of Eq. (56). The off-diagonal blocks and $G^{\rm full}_{QQ}$ retain their Feshbach form.

With this convention, the cross section takes the form

$$\begin{aligned} \sigma^{(B)} &= -\frac{(2\pi)^4}{\pi\, v_a}\,\mathrm{Im}\,\{\mathcal{I}_{\rm ref} + \mathcal{I}_P + \mathcal{I}_{PQ} + \mathcal{I}_{{\rm coup},P} \\ &\quad + \mathcal{I}_{{\rm coup},PQ} + \mathcal{I}_Q\}, \\ \mathcal{I}_{\rm ref} &= \left\langle\rho^{(B,\rm ref)}_i\middle|G^{\rm opt}_{PP}\middle|\rho^{(B,\rm ref)}_i\right\rangle, \\ \mathcal{I}_P &= \left\langle\rho^{(B,\rm ref)}_i\middle|G^{\rm opt}_{PP}\middle|\rho^{(B,\rm coup),P}_i\right\rangle + \left\langle\rho^{(B,\rm coup),P}_i\middle|G^{\rm opt}_{PP}\middle|\rho^{(B,\rm ref)}_i\right\rangle, \\ \mathcal{I}_{PQ} &= \left\langle\rho^{(B,\rm ref)}_i\middle|G_{PQ}\middle|\rho^{(B,\rm coup),Q}_i\right\rangle + \left\langle\rho^{(B,\rm coup),Q}_i\middle|G_{QP}\middle|\rho^{(B,\rm ref)}_i\right\rangle, \\ \mathcal{I}_{{\rm coup},P} &= \left\langle\rho^{(B,\rm coup),P}_i\middle|G^{\rm opt}_{PP}\middle|\rho^{(B,\rm coup),P}_i\right\rangle, \\ \mathcal{I}_{{\rm coup},PQ} &= \left\langle\rho^{(B,\rm coup),P}_i\middle|G_{PQ}\middle|\rho^{(B,\rm coup),Q}_i\right\rangle \\ &\quad + \left\langle\rho^{(B,\rm coup),Q}_i\middle|G_{QP}\middle|\rho^{(B,\rm coup),P}_i\right\rangle, \\ \mathcal{I}_Q &= \left\langle\rho^{(B,\rm coup),Q}_i\middle|G^{\rm full}_{QQ}\middle|\rho^{(B,\rm coup),Q}_i\right\rangle. \end{aligned} \quad (63)$$

The sources on either side of each propagator select its target-space block. The reference source and $\rho_i^{(B,\mathrm{coup}),P}$ lie in P, while $\rho_i^{(B,\mathrm{coup}),Q}$ lies in Q.

The term $\mathcal{I}_{\mathrm{ref}}$ is the CFH reference-channel contribution whose absorptive reduction gives the $W_j + W_k + W_{3B}$ form of Eq. (59); the remaining $\mathcal{I}$ terms are generated by the explicit target-coupling source. The intra-P term $\mathcal{I}_P$ vanishes identically when U_{iA} is chosen as the elastic folding potential, and in general represents the renormalization freedom already discussed in Ref. [33]. The terms involving G_{PQ} and G_{QP} are interferences between the elastic $jk+A$ channel space and target-excited $jk+A^*$ channels, coupled through $V_{jA}+V_{kA}$ at the propagator level and through QV_{iA} at the source level. As in Eq. (30), these interferences are written as explicit PQ and QP pairs before the final imaginary part is taken; replacing them by 2 Re would only be justified after a Hermitian absorptive-kernel reduction. The term $\mathcal{I}_Q$ is the additional target-excited contribution: absorption into $jk+A^*$ continua driven by the explicit target-coupling source. Because it contains G_{QQ}^{full}, it includes both direct propagation in the Q space and the induced $Q\to P\to Q$ rescattering shown in Eq. (62). The separation between $\rho_i^{(B,\mathrm{ref})}$ and $\rho_i^{(B,\mathrm{coup}),P}$ depends on the reference potential U_{iA}, so changes of U_{iA} reshuffle strength among the P-sector terms. Before optical reduction, the sum of the source-kernel matrix elements recovers Eq. (45). With G_{PP}^{exact} replaced by G_{PP}^{opt}, the sum recovers the DWBA expression within that optical approximation. The Q-sector assignment is fixed here by the double-counting rule: the detected-particle transition $QV_{iA}P$ is a source term, while the unresolved-fragment transitions $QV_{jA}P$ and $QV_{kA}P$ remain inside the Feshbach kernels.

For the direct g_Q contribution, I expand the target space as $Q=\sum_{A'\neq 0}|\Phi_{A'}\rangle\langle\Phi_{A'}|$.

Here and below, the $|\Phi_{A'}\rangle$ are the particle-number-conserving eigenstates of the target internal Hamiltonian H_A; the consequence of this choice for pairing-correlated targets is stated in Sec. VIII. For a given excited target state $|\Phi_{A'}\rangle$, the restricted projector $Q_{A'}=|\Phi_{A'}\rangle\langle\Phi_{A'}|$ satisfies $Q_{A'}g_QQ_{A'}\equiv\tilde{G}_{jk}^{(A')}$, where $\tilde{G}_{jk}^{(A')}$ is a three-body jk relative-motion resolvent at the shifted energy $E_{jk}^{(A')}=E_{jkA}-\omega_{A'}$, with $\omega_{A'}\equiv E_{A'}-E_A$ being the excitation energy of the target state $|\Phi_{A'}\rangle$ above the ground state (which equals $E_{A'}$ under the global $E_A=0$ convention of Sec. II). The resolvent $\tilde{G}_{jk}^{(A')}$ is built from the target-excited effective Hamiltonian

$$\begin{aligned}\tilde{\mathcal{H}}_{jk}^{(A')} &= h_j+h_k+K_j+K_k+V_{jk}\\ &\quad+\tilde{U}_j^{(A')}(\mathbf{r}_j)+\tilde{U}_k^{(A')}(\mathbf{r}_k)+\tilde{\mathcal{V}}_{3B}^{(A')},\end{aligned}\tag{64}$$

where the optical interactions are now defined relative to $|\Phi_{A'}\rangle$. A second Feshbach projection onto $Q_\perp^{(A')}=Q-|\Phi_{A'}\rangle\langle\Phi_{A'}|$ gives

$$\begin{aligned}\tilde{U}_j^{(A')}(\mathbf{r}_j) &= \langle\Phi_{A'}|V_{jA}|\Phi_{A'}\rangle\\ &\quad+\langle\Phi_{A'}|V_{jA}\,Q_\perp^{(A')}\,\bar{G}_{Q_\perp}^{(A')}\,Q_\perp^{(A')}\,V_{jA}|\Phi_{A'}\rangle,\\ \tilde{\mathcal{V}}_{3B}^{(A')} &= \langle\Phi_{A'}|V_{jA}\,Q_\perp^{(A')}\,\bar{G}_{Q_\perp}^{(A')}\,Q_\perp^{(A')}\,V_{kA}|\Phi_{A'}\rangle\\ &\quad+(j\leftrightarrow k),\end{aligned}\tag{65}$$

and analogously for $\tilde{U}_k^{(A')}$, where $\bar{G}_{Q_\perp}^{(A')}$ is the resolvent on the $Q_\perp^{(A')}$ subspace, i.e., states whose target component is neither $|\Phi_A\rangle$ nor $|\Phi_{A'}\rangle$ but any other target eigenstate (whether energetically above or below $|\Phi_{A'}\rangle$). Excluding both $|\Phi_A\rangle$ and $|\Phi_{A'}\rangle$ from $Q_\perp^{(A')}$ separates these polarization terms from those of the ground-state block.

I now drop the matrix elements $\langle\Phi_{A''}|g_Q|\Phi_{A'}\rangle$ with $A''\neq A'$. This approximation removes propagation between different target labels at the two ends of g_Q. Each retained diagonal block still contains excursions through other excited states and back to A'. The interaction matrix elements $\langle\Phi_{A'}|V_{jA}|\Phi_{A''}\rangle$ and $\langle\Phi_{A'}|V_{kA}|\Phi_{A''}\rangle$ $(A''\neq A,A')$ therefore remain in the polarization terms of Eq. (65). Keeping the off-diagonal blocks of g_Q would instead require coupled propagation in the excited-target basis. Representing the polarization terms within each retained block by optical interactions is a further approximation to the nested Feshbach expressions in Eq. (65); these optical interactions are diagonal in the intermediate target label.

Under the diagonal approximation,

$$g_Q\approx\sum_{A'\neq 0}|\Phi_{A'}\rangle\tilde{G}_{jk}^{(A')}\langle\Phi_{A'}|,\tag{66}$$

and

$$\begin{aligned}-\pi^{-1}\mathrm{Im}\,\tilde{G}_{jk}^{(A')} &\approx \big|\tilde{\chi}_{jk}^{(A',+)}\big\rangle\big\langle\tilde{\chi}_{jk}^{(A',+)}\big|_{\text{on-shell}}\\ &\quad-\pi^{-1}\,\tilde{G}_{jk}^{(A')\dagger}\,\tilde{W}_{\mathrm{CFH}}^{(A')}\,\tilde{G}_{jk}^{(A')},\end{aligned}\tag{67}$$

with $\tilde{W}_{\mathrm{CFH}}^{(A')}\equiv\tilde{W}_j^{(A')}+\tilde{W}_k^{(A')}+\tilde{W}_{3B}^{(A')}$, in the same two-potential form as Eqs. (31) and (58); here $|\tilde{\chi}_{jk}^{(A',+)}\rangle$ is the three-body jk elastic-scattering wave function in the target field of the excited state $|\Phi_{A'}\rangle$ and energy $E_{jk}^{(A')}$, and $\tilde{W}_{j,k}^{(A')}$, $\tilde{W}_{3B}^{(A')}$ are the target-excited analogs of the CFH absorption operators, constructed from $\tilde{U}_j^{(A')}$, $\tilde{U}_k^{(A')}$ and from the cross term $\tilde{\mathcal{V}}_{3B}^{(A')}$ with propagation in $Q_\perp^{(A')}$. Substituting Eq. (66) into the direct g_Q part of the last line of Eq. (63) gives, in the diagonal-intermediate-target-states approximation,

$$\begin{aligned}\big\langle\rho_i^{(B,\mathrm{coup}),Q}\big|g_Q\big|\rho_i^{(B,\mathrm{coup}),Q}\big\rangle &\approx \sum_{A'\neq 0}\big\langle\hat{\rho}_{jk}^{(A')}\big|\tilde{G}_{jk}^{(A')}\big|\hat{\rho}_{jk}^{(A')}\big\rangle,\\ \hat{\rho}_{jk}^{(A')} &= \big\langle\Phi_{A'}\big|\rho_i^{(B,\mathrm{coup}),Q}\big\rangle.\end{aligned}\tag{68}$$

To obtain the direct-g_Q contribution to the cross section, multiply the imaginary part of Eq. (68) by $-(2\pi)^4(\pi v_a)^{-1}$. Applying the two-potential identity (67) to each $\tilde{G}_{jk}^{(A')}$ then splits this into target-excited NEB and EBU densities,

$$\begin{aligned}&-\pi^{-1}\sum_{A'\neq 0}\mathrm{Im}\big\langle\hat{\rho}_{jk}^{(A')}\big|\tilde{G}_{jk}^{(A')}\big|\hat{\rho}_{jk}^{(A')}\big\rangle\\ &\quad\approx\sum_{A'\neq 0}\Big\{\big|\big\langle\tilde{\chi}_{jk}^{(A',+)}\big|\hat{\rho}_{jk}^{(A')}\big\rangle\big|^2\\ &\qquad-\pi^{-1}\big\langle\hat{\rho}_{jk}^{(A')}\big|\tilde{G}_{jk}^{(A')\dagger}\,\tilde{W}_{\mathrm{CFH}}^{(A')}\,\tilde{G}_{jk}^{(A')}\big|\hat{\rho}_{jk}^{(A')}\big\rangle\Big\},\end{aligned}\tag{69}$$

so that the direct target-excited absorption contributes a sum of CFH-like NEB structures indexed by the populated target

excitation A', each carrying its own $\tilde{W}_j^{(A')} + \tilde{W}_k^{(A')} + \tilde{W}_{3B}^{(A')}$ decomposition, accompanied by an EBU contribution into the jk elastic continuum on top of $|\Phi_{A'}\rangle$. For targets with strong low-lying collective modes, leading contributions are expected from surface dipole and quadrupole excitations reached by $QV_{iA}P$ acting on $|\Phi_A\rangle$. The $\tilde{U}_j^{(A')}$, $\tilde{U}_k^{(A')}$ are the shifted-energy analogs of nucleon-nucleus optical interactions in the target-excited sector and must be distinguished from the ground-state optical potentials entering the CFH reference term to avoid double counting the same target-polarization strength. In the CFH reference limit $V_{iA} \to U_{iA}$, only $\mathcal{I}_{\rm ref}$ survives, recovering Eq. (59).

For a structureless detected particle i, the operator $V_{iA} - U_{iA}$ coincides with the familiar nonelastic (target-excitation) correction of ordinary DWBA; the $\rho_i^{(B,\mathrm{coup}),Q}$ term in Eq. (63) then corresponds to the conventional target-polarization correction, and the extent to which Eq. (63) contains physics beyond the existing DWBA practice is confined to how this correction interferes with the W_{3B} channel of CFH. For a composite detected particle ($i = d$, α, or ^{8}Be), the $\boldsymbol{\zeta}_i$ dependence of V_{iA} brings in the multipole structure of Ref. [33] and Eq. (93) below, which can be substantial. The reduction in Eq. (68) applies only to the direct g_Q contribution. Off-diagonal propagation between excited target labels and the associated source nonorthogonality require the full coupled-channel treatment; the nested polarization inside each diagonal block remains, as in Eq. (65).

V. PRIOR FORMS AND POST-PRIOR RELATIONS

The entrance-channel Hamiltonian for the four-body system is

$$H_{\rm ent} = H_A + K_a + U_a + h_a, \tag{70}$$

where K_a is the $a-A$ center-of-mass kinetic energy, U_a is the entrance-channel optical potential, and h_a is the internal Hamiltonian of the projectile, $h_a = h_i + h_j + h_k + K_\zeta + K_y + V_{ij} + V_{ik} + V_{jk}$. The prior-form residual interaction is common to both partitions,

$$V_{\rm prior} = H - H_{\rm ent} = V_{iA} + V_{jA} + V_{kA} - U_a. \tag{71}$$

The same residual interaction therefore generates both prior sources. The post-prior identity is established below for pair detection and, for single-particle detection, within the CFH target-ground-state optical reduction. The target-excited terms are retained through the Feshbach decomposition; no separate post-prior identity is proved for them. These prior forms provide the starting point for calculations where the corresponding IAV and UT convergence properties apply.

For Partition A, the prior-form T-matrix is $T_{\alpha,c}^{\prime(A)} = \langle \chi_b^{(-)} \phi_\alpha \, \Psi_{kA}^c | \, V_{\rm prior} \, | \chi_a^{(+)} \Phi_a \Phi_A \rangle$, and the prior source is

$$\left| \rho_\alpha^{\prime(A)} \right\rangle = \langle \phi_\alpha \, \chi_b^{(-)} | \, V_{\rm prior} \, | \chi_a^{(+)} \Phi_a \Phi_A \rangle. \tag{72}$$

In the reduced single-channel limit, the prior source decomposes as $\tilde{\rho}_\alpha^{\prime(A)} = \tilde{\rho}_\alpha^{\prime(A,\mathrm{UT})} + \tilde{\rho}_\alpha^{\prime(A,\mathrm{coup})}$, where the UT source involves $U_k + U_{bA} - U_a$ and the explicit coupling source involves $V_{bA} - U_{bA}$, the same operator as in the post form. The UT NEB cross section in prior form is

$$\sigma_\alpha^{(A,\mathrm{UT})} = -\frac{(2\pi)^4}{\pi \, v_a} \left\langle \tilde{\rho}_\alpha^{\prime(A,\mathrm{UT})} \middle| G_k^\dagger \, W_k \, G_k \middle| \tilde{\rho}_\alpha^{\prime(A,\mathrm{UT})} \right\rangle, \tag{73}$$

and the nonorthogonality (NO) correction, i.e., the Hussein-McVoy term, and the interference (IN) correction take the standard form [10,34]. The nonorthogonality overlap, projected onto the pair's internal state, is

$$\psi_\alpha^{(A,\mathrm{NO})}(\mathbf{r}_k) = \int d\mathbf{r}_b \, d\boldsymbol{\zeta} \; \chi_b^{(-)*} \, \phi_\alpha^* \, \chi_a^{(+)} \, \Phi_a, \tag{74}$$

which does not involve any interaction and is independent of the choice of reference interaction.

For Partition B, the prior source is

$$\left| \rho_i^{\prime(B)} \right\rangle = \langle \chi_i^{(-)} | \, V_{\rm prior} \, | \chi_a^{(+)} \Phi_a \Phi_A \rangle, \tag{75}$$

which decomposes into UT and explicit-coupling parts, with the UT source built from $U_j + U_k + U_{iA} - U_a$ and the explicit coupling source from $V_{iA} - U_{iA}$. As in the CFH reduction of Sec. IV, the target couplings of the unresolved constituents j and k are assigned to the three-body propagator and to its absorptive kernel, rather than to an additional source term.

The difference between post and prior residual interactions is, for Partition A,

$$V_{\rm post}^{(A)} - V_{\rm prior} = V_{bk} - V_{kA} + U_a - U_b, \tag{76}$$

where V_{bA} cancels exactly, just as in the two-body case [6,33]. For Partition B,

$$V_{\rm post}^{(B)} - V_{\rm prior} = V_{ij} + V_{ik} - V_{jA} - V_{kA} + U_a - U_i, \tag{77}$$

where V_{iA} cancels exactly. The explicit target coupling of the detected pair in Partition A, $V_{bA} - U_{bA}$, and of the detected particle in Partition B, $V_{iA} - U_{iA}$, each enters the post and prior sources identically and therefore drops out of their difference. Equations (76) and (77) are operator identities valid on any state; the corresponding matrix-element identity $T^{\rm post} = T^{\rm prior}$ requires in addition that the entrance and exit channel functions satisfy their respective Schrödinger equations up to surface terms. For Partition A, the entrance state $\chi_a^{(+)}\Phi_a\Phi_A$ and the exit state $\chi_b^{(-)}\phi_\alpha\Psi_{kA}^c$ are governed by two-body asymptotic Schrödinger equations. For charged systems, the long-range Coulomb pieces must first be absorbed consistently into the channel distortions, so that the operators entering the post-prior difference are short-ranged. Under that standard assumption, the surface terms vanish, and the full-space DWBA post-prior relation for Partition A follows by the same algebra as in the two-body derivation [6,33,34]. For Partition B the analogous full-space step requires Faddeev asymptotics for the three-body exit state Ψ_{jkA}^c and is left out of scope, as discussed below; the reduced identity used in this paper is established only at the CFH-reduced level. The post-prior identity connecting the two representations is, for Partition A in the reduced single-channel limit,

$$G_k \left| \tilde{\rho}_\alpha^{\prime(A,\mathrm{UT})} \right\rangle = G_k \left| \tilde{\rho}_\alpha^{(A,\mathrm{ref})} \right\rangle - \left| \psi_\alpha^{(A,\mathrm{NO})} \right\rangle, \tag{78}$$

and the explicit coupling sources are identical in both forms, $\tilde{\rho}_\alpha^{\prime(A,\mathrm{coup})} = \tilde{\rho}_\alpha^{(A,\mathrm{coup})}$. Substituting Eq. (78) into the prior-form NEB cross section produces the Hussein-McVoy decomposition, extended here to include the explicit coupling source.

Defining the auxiliary kets $|\Lambda_\alpha^{\rm UT}\rangle = G_k|\tilde\rho_\alpha'^{(A,{\rm UT})}\rangle$, $|\Lambda_\alpha^{\rm NO}\rangle = |\psi_\alpha^{(A,{\rm NO})}\rangle$, and $|\Lambda_\alpha^{\rm coup}\rangle = G_k|\tilde\rho_\alpha^{(A,{\rm coup})}\rangle$, the NEB cross section splits into five terms,

$$\begin{aligned}
\sigma_\alpha^{(A,{\rm NEB})} &= \sigma_\alpha^{\rm UT} + \sigma_\alpha^{\rm NO} + \sigma_\alpha^{\rm IN} + \sigma_\alpha^{\rm coup,pure} + \sigma_\alpha^{\rm coup,IN},\\
\sigma_\alpha^{\rm UT} &= -\frac{(2\pi)^4}{\pi v_a}\langle\Lambda_\alpha^{\rm UT}|W_k|\Lambda_\alpha^{\rm UT}\rangle,\\
\sigma_\alpha^{\rm NO} &= -\frac{(2\pi)^4}{\pi v_a}\langle\Lambda_\alpha^{\rm NO}|W_k|\Lambda_\alpha^{\rm NO}\rangle,\\
\sigma_\alpha^{\rm IN} &= -\frac{2(2\pi)^4}{\pi v_a}\,{\rm Re}\langle\Lambda_\alpha^{\rm UT}|W_k|\Lambda_\alpha^{\rm NO}\rangle,\\
\sigma_\alpha^{\rm coup,pure} &= -\frac{(2\pi)^4}{\pi v_a}\langle\Lambda_\alpha^{\rm coup}|W_k|\Lambda_\alpha^{\rm coup}\rangle,\\
\sigma_\alpha^{\rm coup,IN} &= -\frac{2(2\pi)^4}{\pi v_a}\,{\rm Re}\langle\Lambda_\alpha^{\rm UT}+\Lambda_\alpha^{\rm NO}|W_k|\Lambda_\alpha^{\rm coup}\rangle. \qquad (79)
\end{aligned}$$

The UT, NO, and IN terms are the usual Hussein-McVoy contributions [10,11,34], now projected onto ϕ_α. The overlap in Eq. (74) retains the full Φ_a. The last two terms give the square of the explicit-coupling amplitude and its interference with the UT and NO amplitudes. Both are needed even though the explicit source is identical in post and prior forms.

With the convention $W_k \leqslant 0$, the quadratic UT and NO terms are non-negative, while the IN piece can have either sign. The latter is known to be numerically important near and above the Coulomb barrier [10,34]. Substituting Eq. (79) back into the post form verifies post-prior equivalence for the reduced single-channel NEB cross section in Partition A, including the reference and explicit-coupling source contributions at that level.

For Partition B, a full-space post-prior identity at the level of the unreduced DWBA matrix element would require Faddeev-asymptotic exit states with Dollard-regularized Coulomb phases on each two-fragment partition of $jk+A$, together with a controlled treatment of the connected breakup hyperplane. This is beyond the scope of the present DWBA-level derivation; the analogous problem in the two-body-projectile limit is discussed in Ref. [23]. No result below depends on a full-space identity for Partition B. The target-excited terms retain the Feshbach decomposition and the source and propagator assignments of Sec. IV.

For the target–ground-state sector, I use the energy-independent CFH optical Hamiltonian to derive the counterpart of Eq. (78).

The projected post- and prior-form sources are $\hat\rho^{(B,{\rm ref})} = \langle\Phi_A|\rho_i^{(B,{\rm ref})}\rangle$ and $\hat\rho'^{(B,{\rm UT})} = \langle\Phi_A|\rho_i'^{(B,{\rm UT})}\rangle$, with $\rho_i'^{(B,{\rm UT})}$ built from $U_j+U_k+U_{iA}-U_a$ in the prior residual. The reduced exit-channel Hamiltonian on the P sector is $\bar H_{\rm exit}^{(B,P)} = h_i + K_i + U_i + \mathcal{H}_{PP}^{\rm opt}$, with $\mathcal{H}_{PP}^{\rm opt} = h_j + h_k + K_j + K_k + V_{jk} + U_j + U_k + \mathcal{V}_{3B}$ from Eq. (56), so that the optical $G_{PP}^{\rm opt}$ obeys $(E-\bar H_{\rm exit}^{(B,P)})G_{PP}^{\rm opt} = \mathbf{1}$ in the P-sector relative-motion Hilbert space. The interaction $\mathcal{V}_{3B}$ is included in this Hamiltonian and in $G_{PP}^{\rm opt}$. As in the optical reduction of Sec. IV, $\Delta\mathcal{H}_{PP}$ is treated as energy-independent. Retaining its energy dependence would introduce $\partial_E\Delta\mathcal{H}_{PP}$ terms, which are not treated in this derivation.

Applying $\langle\Phi_A|$ to the post and prior sources turns the microscopic operators V_{jA}, V_{kA}, V_{iA} into their ground-state expectations, absorbed into the optical $U_j+U_k+U_{iA}$, while the $V_{ij}+V_{ik}$ part of the post-form residual is target-trivial. Acting $G_{PP}^{\rm opt}$ from the left and using the on-shell Schrödinger identity for the asymptotic exit state $|\chi_i^{(-)}\rangle\otimes|\chi_{jk}^{(+)}\rangle\,\Phi_A$, exactly as in the two-body derivation of Eq. (78), one obtains

$$G_{PP}^{\rm opt}\,|\hat\rho'^{(B,{\rm UT})}\rangle = G_{PP}^{\rm opt}\,|\hat\rho^{(B,{\rm ref})}\rangle - |\hat\psi_{3B}^{(B,{\rm NO})}\rangle, \qquad (80)$$

where the three-body analog of the nonorthogonality overlap is

$$\begin{aligned}
\hat\psi_{3B}^{(B,{\rm NO})}(\mathbf{r}_j,\mathbf{r}_k) = \int d\mathbf{r}_i\,d\xi\;&\chi_i^{(-)*}(\mathbf{r}_i)\,\Phi_A^*(\xi)\\
&\times\chi_a^{(+)}(\mathbf{R}(\mathbf{r}_i,\mathbf{r}_j,\mathbf{r}_k))\\
&\times\Phi_a(\boldsymbol{\zeta}'(\mathbf{r}_j,\mathbf{r}_k),\,\mathbf{y}'(\mathbf{r}_i,\mathbf{r}_j,\mathbf{r}_k))\,\Phi_A(\xi), \qquad (81)
\end{aligned}$$

where the dependence of the projectile Jacobi coordinates and of the entrance distorted-wave argument on $(\mathbf{r}_i,\mathbf{r}_j,\mathbf{r}_k)$ is given by the kinematic relations of Sec. II. The $\mathbf{r}_i$ integration projects out the detected-particle coordinate; target-state orthonormality $\langle\Phi_A|\Phi_A\rangle = 1$ removes ξ. The result is a function of the unresolved-pair coordinates $(\mathbf{r}_j,\mathbf{r}_k)$ alone, equivalently a function of the (jk)-internal Jacobi coordinate $\boldsymbol{\zeta}'' = \mathbf{r}_j - \mathbf{r}_k$ and the (jk)-c.m.–target coordinate $\mathbf{r}_{(jk)A} = (m_j\mathbf{r}_j + m_k\mathbf{r}_k)/(m_j+m_k)$ of Eq. (8). Thus $\hat\psi_{3B}^{(B,{\rm NO})}$ is the entrance-exit overlap after projection onto the target ground state and the detected particle. Equation (80) has the form of Eq. (78), with $G_k \to G_{PP}^{\rm opt}$ and $\psi_\alpha^{(A,{\rm NO})} \to \hat\psi_{3B}^{(B,{\rm NO})}$. This is not an exact Feshbach-level identity; it retains the optical and Coulomb assumptions used above.

Within this ground-state-sector reduction, the detected-particle explicit source operator is the same in post and prior forms, $\hat\rho'^{(B,{\rm coup}),P} = \hat\rho^{(B,{\rm coup}),P}$. Substituting Eq. (80) into the prior-form reduced cross section produces a Hussein-McVoy-type decomposition with five explicit terms, analogous to Eq. (79). Defining the auxiliary kets $|\Lambda^{\rm UT,B}\rangle = G_{PP}^{\rm opt}|\hat\rho'^{(B,{\rm UT})}\rangle$, $|\Lambda^{\rm NO,B}\rangle = |\hat\psi_{3B}^{(B,{\rm NO})}\rangle$, and $|\Lambda^{\rm coup,B}\rangle = G_{PP}^{\rm opt}|\hat\rho^{(B,{\rm coup}),P}\rangle$, and writing $W_{\rm CFH} \equiv W_j + W_k + W_{3B}$, the ground-state-sector NEB cross section reads

$$\begin{aligned}
\sigma_{\rm CFH}^{(B,{\rm NEB})} &= \sigma^{\rm UT,B} + \sigma^{\rm NO,B} + \sigma^{\rm IN,B}\\
&\quad + \sigma^{\rm coup,pure,B} + \sigma^{\rm coup,IN,B},\\
\sigma^{\rm UT,B} &= -\frac{(2\pi)^4}{\pi v_a}\langle\Lambda^{\rm UT,B}|W_{\rm CFH}|\Lambda^{\rm UT,B}\rangle,\\
\sigma^{\rm NO,B} &= -\frac{(2\pi)^4}{\pi v_a}\langle\Lambda^{\rm NO,B}|W_{\rm CFH}|\Lambda^{\rm NO,B}\rangle,\\
\sigma^{\rm IN,B} &= -\frac{2(2\pi)^4}{\pi v_a}\,{\rm Re}\langle\Lambda^{\rm UT,B}|W_{\rm CFH}|\Lambda^{\rm NO,B}\rangle,\\
\sigma^{\rm coup,pure,B} &= -\frac{(2\pi)^4}{\pi v_a}\langle\Lambda^{\rm coup,B}|W_{\rm CFH}|\Lambda^{\rm coup,B}\rangle,\\
\sigma^{\rm coup,IN,B} &= -\frac{2(2\pi)^4}{\pi v_a}\,{\rm Re}\langle\Lambda^{\rm UT,B}+\Lambda^{\rm NO,B}|W_{\rm CFH}|\Lambda^{\rm coup,B}\rangle. \qquad (82)
\end{aligned}$$

The two-body W_k in Eq. (79) is replaced here by $W_{\rm CFH} = W_j + W_k + W_{3B}$, and the overlap by $\hat{\psi}_{3B}^{(B,\rm NO)}$ [Eq. (81)]. If $W_{\rm CFH}$ is negative semidefinite, the UT, NO, and pure-coupling terms are non-negative, while both interferences may have either sign. This sign condition holds for the microscopic Feshbach kernel and is assumed in its phenomenological replacement as discussed in Sec. IV.

Equation (82) holds in the CFH optical reduction of the target-ground-state sector. The target-excited terms [Eqs. (63)–(68)] remain separate. In the microscopic prior residual, $QV_{\rm prior}P$ contains $QV_{iA}P$, $QV_{jA}P$, and $QV_{kA}P$. The last two already enter the Feshbach kernel of $jk+A$ and cannot be counted again as source corrections. The explicit target-excited source therefore contains $QV_{iA}P$ alone [Eq. (61)]. A post-prior identity for this sector would require the full coupled-channel Q-space metric, with the source and kernel varied together. It is not established by the diagonal-intermediate-states reduction used here.

VI. LIMITING REDUCTIONS AND CONNECTIONS TO EXISTING FORMALISMS

The two-body IAV sum rule, the detected-cluster result of Ref. [33], and CFH [29] follow by restricting the projectile wave function or the target coupling as described below.

For Partition A, replacing the full pair-target interaction by the reference interaction, $V_{bA} \to U_{bA}$, removes the explicit coupling source and leaves $\rho_\alpha^{(A,\rm ref)}$. If the pair $b = (ij)$ is further reduced to a preformed structureless cluster, one recovers the standard two-body IAV with $a = b + k$ and $x = k$. Taking $\alpha = 0$ and absorbing the internal $\boldsymbol{\zeta}$ dependence into cluster interactions turns $V_{bk} + U_{bA} - U_b$ into $V_{bx} + U_{bA} - U_b$, with $x = k$. For ^{6}Li, V_{bx} is the folded interaction $\langle \phi_d | V_{n\alpha} + V_{p\alpha} | \phi_d \rangle_\zeta$, equal to $\mathcal{F}_{dd}$ in Eq. (97).

Under both operations, the source

$$\left|\rho_\alpha^{(A,\rm ref)}\right\rangle \xrightarrow{\rm cluster} \langle \chi_b^{(-)} | (V_{bx} + U_{bA} - U_b) | \chi_a^{(+)} \varphi_a \rangle \equiv |\rho_{\text{2-body IAV}}\rangle, \tag{83}$$

and Eq. (34) becomes the standard IAV [6] master formula with the two-body projectile wave function $\varphi_a(\mathbf{r}_{bk}) = f(\mathbf{y})|_{\rm cluster\ limit}$, where $\mathbf{r}_{bk} = -\mathbf{y}$ is the pair-to-k relative coordinate in the two-body convention of Refs. [6,33].

The connection to the two-fragment detected-cluster formalism of Ref. [33] is obtained by a different reduction. If the projectile a happens to have a dominant two-body cluster structure $a \approx b + k$ with b a bound state of (ij), then the three-body wave function factorizes approximately as $\Phi_a(\boldsymbol{\zeta}, \mathbf{y}) \approx \phi_0(\boldsymbol{\zeta}) f(\mathbf{y})$, where $f(\mathbf{y})$ describes the $b-k$ relative motion. In this limit, the pair projection $\langle \phi_0 | \Phi_a \rangle_\zeta = f(\mathbf{y})$ reduces to the two-body wave function, and the four-body source reduces to the two-fragment source of Ref. [33] with $\varphi_a(\mathbf{r}_{bk}, \boldsymbol{\zeta}) = \Phi_a(\boldsymbol{\zeta}, \mathbf{y})$. In the full source, the detected pair is projected from the three-body projectile wave function rather than prescribed as an entrance-channel cluster, and its internal states are resolved by the $\boldsymbol{\zeta}$-dependent operators. Comparing the source calculated with the full Φ_a and with $\phi_0 f$ measures the cluster approximation at the amplitude level.

For Partition B in the CFH reference limit ($V_{iA} \to U_{iA}$), the result reduces to Eq. (48), which is the starting point of the CFH formalism. The CFH reference source involves the projectile wave function Φ_a projected onto the distorted waves $\chi_i^{(-)}$ and $\chi_a^{(+)}$, and the three-body resolvent G_{jkA} is treated within the optical-model framework for j and k separately. The full four-body source also contains $V_{iA} - U_{iA}$. For a composite detected particle, this term contains the finite-size tidal coupling of i to the target; for a structureless detected particle, it is the difference between the microscopic and optical-model $i - A$ interactions. For structureless i, the target-excitation part can be expanded in multipoles:

$$V_{iA}(\mathbf{r}_{iA}, \xi) = \sum_{\lambda\mu} v_\lambda(r_{iA})\, T_\mu^\lambda(\xi) Y_\mu^{\lambda *}(\hat{\mathbf{r}}_{iA}), \tag{84}$$

where $\mathbf{r}_{iA}$ is the exact $i - A$ separation of Eq. (8), evaluated inside the Partition-B source integrals at $\mathbf{r}_{iA} = \mathbf{r}_i + \frac{m_j + m_k}{m_j + m_k + m_A} \mathbf{r}_{(jk)A}$ with $\mathbf{r}_{(jk)A}$ determined by the source output variables while the integration runs over the Jacobi coordinate $\mathbf{r}_i$; $T_\mu^\lambda(\xi)$ are target multipole operators (whose isoscalar or isovector character depends on the underlying NN-effective interaction and on whether neutron-proton asymmetries of the target one-body density are resolved; the even-λ pieces typically receive sizable isoscalar contributions and the odd-λ pieces receive isovector contributions, but the exact decomposition is target- and energy-specific) and $v_\lambda(r_{iA})$ are radial form factors obtained, e.g., by folding a t- or g-matrix with the target one-body density. With these definitions one has

$$\begin{aligned} \langle \Phi_A | V_{iA} | \Phi_A \rangle &= v_0(r_{iA})\, \langle \Phi_A | T_0^0 | \Phi_A \rangle, \\ \langle \Phi_{A'} | V_{iA} | \Phi_A \rangle &= \sum_{\lambda\mu} v_\lambda(r_{iA}) Y_\mu^{\lambda *}(\hat{\mathbf{r}}_{iA})\, \langle \Phi_{A'} | T_\mu^\lambda | \Phi_A \rangle, \end{aligned} \tag{85}$$

the first of which equals U_{iA} (up to the polarization renormalization already absorbed in the phenomenological optical potential) and the second controls $QV_{iA}P$. The target-transition matrix elements $\langle \Phi_{A'} | T_\mu^\lambda | \Phi_A \rangle$ are exactly the reduced matrix elements entering inelastic scattering of i from the target; they are known from structure calculations or from fits to inelastic (p, p'), (α, α'), etc. data and are quantitatively controlled by low-lying collective excitations of A (surface dipole, quadrupole, octupole). The explicit coupling source in the structureless case therefore inherits the same inelastic form factors that dominate ordinary inelastic scattering, and the Q-sector reduction of Eq. (68) becomes, for each low-lying A',

$$\hat{\rho}_{jk}^{(A')}(\mathbf{r}_j, \mathbf{r}_k) = \sum_{\lambda\mu} \langle \Phi_{A'} | T_\mu^\lambda | \Phi_A \rangle\, \mathcal{R}_{\lambda\mu}(\mathbf{r}_j, \mathbf{r}_k), \tag{86}$$

with $\mathcal{R}_{\lambda\mu}(\mathbf{r}_j, \mathbf{r}_k) = \langle \chi_i^{(-)} |\, v_\lambda(r_{iA}) Y_\mu^{\lambda *}(\hat{\mathbf{r}}_{iA}) | \chi_a^{(+)} \Phi_a \rangle$ a multipole form factor in the three-body relative-motion space. The source therefore requires the inelastic form factors v_λ and target transition strengths $B(E\lambda; 0 \to A') \equiv (2J_0 + 1)^{-1} |\langle \Phi_{A'} \| T^\lambda \| \Phi_A \rangle|^2$ that determine single-particle inelastic scattering, folded with the three-body projectile structure.

Table II collects the cluster and optical reductions. The target-excited kernels apply to the direct g_Q contribution under

TABLE II. Reduction of the present four-body framework to existing formalisms in various limits.

Limit	Recovered or obtained formalism
A, $V_{bA} \to U_{bA}$, $\Phi_a \to \phi_0 f$	Two-body IAV [6]
A, full V_{bA}, $\Phi_a \to \phi_0 f$	Two-fragment detected-cluster result [33]
A, full V_{bA}, full Φ_a	This work, pair detection
B, $V_{iA} \to U_{iA}$, reduced G_{jkA}	CFH [29]
B, full V_{iA}, unreduced G_{jkA}	This work, single-particle starting point
B, full V_{iA}, diagonal direct-g_Q reduction	Target-excited CFH-like kernel for the direct g_Q part of $\mathcal{I}_Q$; $\mathcal{I}_{PQ}$, $\mathcal{I}_{\mathrm{coup},PQ}$ interferences and the induced $Q \to P \to Q$ piece left unreduced

the diagonal approximation; the interference terms remain in Feshbach form.

VII. APPLICATION TO ^{6}Li REACTIONS

For $^6\mathrm{Li} + A \to d + (\alpha + A)^*$, I use the three-body description $^6\mathrm{Li} = \alpha + n + p$, with the detected pair $b = d = (np)$ and unresolved fragment $k = \alpha$. Spin labels were suppressed in Secs. II–VI. The deuteron has $J^\pi = 1^+$ and $L = 0, 2$ components. The multipole expressions below use its M_J-averaged density [Eqs. (92), (93)]; the cross sections require the spin sums and averages appropriate to the measurement.

The identification of the constituents is $i = n$, $j = p$, $k = \alpha$, so the pair internal coordinate $\boldsymbol{\zeta} = \mathbf{r}_n - \mathbf{r}_p$ is the $n - p$ relative coordinate and $\mathbf{y} = \mathbf{r}_\alpha - (m_n\mathbf{r}_n + m_p\mathbf{r}_p)/(m_n + m_p)$ is the $\alpha - d$ relative coordinate. The internal Hamiltonian of the pair is $h_b = K_\zeta + V_{np}$, with the deuteron ground state $h_b|\phi_d\rangle = -\epsilon_d|\phi_d\rangle$ at binding energy $\epsilon_d = 2.224$ MeV. The three-body projectile wave function $\Phi_a(\boldsymbol{\zeta}, \mathbf{y})$ satisfies

$$(K_\zeta + K_y + V_{np} + V_{n\alpha} + V_{p\alpha})\,\Phi_a = -\epsilon_a\,\Phi_a, \tag{87}$$

where $\epsilon_a = 3.70$ MeV is the $^6\mathrm{Li} \to \alpha + n + p$ three-body breakup threshold.

The bound deuteron avoids the continuum-pair normalization while retaining sensitivity to its internal structure. I return to an unbound detected pair, ^{8}Be from ^{9}Be, at the end of this section.

The composite interactions [Eqs. (10) and (11)] become

$$V_{bk} = V_{n\alpha} + V_{p\alpha}, \tag{88}$$

$$V_{bA} = V_{nA} + V_{pA}, \tag{89}$$

where, relative to the pair center of mass, the nucleons sit at $\pm\boldsymbol{\zeta}/2$ (for $m_n \approx m_p$), so that their exact separations from the target are $\mathbf{r}_{dA} + \boldsymbol{\zeta}/2$ (neutron) and $\mathbf{r}_{dA} - \boldsymbol{\zeta}/2$ (proton), with $\mathbf{r}_{dA} = \mathbf{r}_b + \frac{m_\alpha}{m_\alpha + m_A}\mathbf{r}_\alpha$ the exact deuteron-target separation [Eq. (8) with $k = \alpha$] and $\mathbf{r}_\alpha$ the α–A relative coordinate. The post-form residual interaction [Eq. (13)] is $V^{(A)}_{\mathrm{post}} = V_{n\alpha} + V_{p\alpha} + V_{nA} + V_{pA} - U_d$.

The reference-channel source for detecting the deuteron [Eq. (19)] has the explicit form

$$\tilde{\rho}^{(\mathrm{ref})}_d(\mathbf{r}_\alpha) = \int d\mathbf{r}_b\, d\boldsymbol{\zeta}\; \chi^{(-)*}_d\, \phi^*_d \times (V_{n\alpha} + V_{p\alpha} + U_{dA} - U_d)\, \chi^{(+)}_a\, \Phi_a(\boldsymbol{\zeta}, \mathbf{y}), \tag{90}$$

where $\mathbf{r}_\alpha$ is the α–A relative coordinate (the variable of the $\alpha + A$ propagator), $\mathbf{y} = \mathbf{r}_\alpha - \mathbf{r}_{dA}$ and the entrance distorted-wave argument $\mathbf{R} = (m_d\mathbf{r}_{dA} + m_\alpha\mathbf{r}_\alpha)/M_a$ follow from the exact relations of Eq. (9), U_{dA} is evaluated at $\mathbf{r}_{dA}$, and the integration runs over the deuteron Jacobi coordinate $\mathbf{r}_b$ and the $n - p$ relative coordinate $\boldsymbol{\zeta}$. $V_{n\alpha}$ and $V_{p\alpha}$ depend on $\boldsymbol{\zeta}$ through the α-nucleon separations, which equal $\mathbf{y} - \boldsymbol{\zeta}/2$ (α relative to n) and $\mathbf{y} + \boldsymbol{\zeta}/2$ (α relative to p) for $m_n \approx m_p$, so the $\boldsymbol{\zeta}$ integration does not factorize even for the reference-channel source.

The explicit pair-target coupling involves the tidal operator $V_{bA} - U_{bA} = V_{nA} + V_{pA} - U_{dA}$. Expanding in $\boldsymbol{\zeta}$ separates the pair's response into dipole, quadrupole, and monopole terms. These also occur in the factorized two-fragment limit of Ref. [33], with mass factors $\eta_{n,p} = m_{n,p}/(m_n + m_p)$. Here they enter the full source through $V_{nA} + V_{pA}$. The pair matrix elements $\langle\phi_d|\boldsymbol{\zeta}^n|\phi_\nu\rangle_\zeta$ are weighted by the three-body amplitudes $a_\nu(\mathbf{y})$. The additional dependence on projectile structure appears in the form factors $\mathcal{F}_{d\nu}$ [Eq. (97)] and the source difference [Eq. (100)] derived below. The off-diagonal terms $\mathcal{F}_{d\nu}$ with $\nu \neq d$ and their contribution to this source difference are absent from the factorized cluster description of Ref. [33].

Using the exact nucleon-target separations $\mathbf{r}_{nA} = \mathbf{r}_{dA} + \eta_p\boldsymbol{\zeta}$, $\mathbf{r}_{pA} = \mathbf{r}_{dA} - \eta_n\boldsymbol{\zeta}$ with $\eta_{n,p} = m_{n,p}/(m_n + m_p)$ and expanding each nucleon-target interaction about the pair-target separation $\mathbf{r}_{dA}$,

$$V_{bA}(\mathbf{r}_{dA}, \boldsymbol{\zeta}) = V_\Sigma(\mathbf{r}_{dA}) + (\eta_p\nabla V_{nA} - \eta_n\nabla V_{pA})\cdot\boldsymbol{\zeta} + \tfrac{1}{2}\big[\eta_p^2\,\partial_a\partial_b V_{nA} + \eta_n^2\,\partial_a\partial_b V_{pA}\big]\zeta^a\zeta^b + O(\zeta^3), \tag{91}$$

where $V_\Sigma = V_{nA} + V_{pA}$ is the isoscalar-like sum and repeated Cartesian indices are summed. Equation (91) is the general-mass expansion; the form factors retain η_n, η_p explicitly so the same expansion applies to any pair $b = (ij)$ with arbitrary mass ratio (e.g., $\alpha\alpha$ in ^{9}Be). Specializing to $m_n \approx m_p$ so that $\eta_n = \eta_p = 1/2$, and introducing the isovector combination $V_\Delta = V_{nA} - V_{pA}$ (dominated by the Lane symmetry term and by the Coulomb difference on charged targets), the linear term collapses to $\frac{1}{2}\boldsymbol{\zeta}\cdot\nabla V_\Delta$ and the quadratic term to $\frac{1}{8}(\boldsymbol{\zeta}\cdot\nabla)^2 V_\Sigma$. Averaging the interaction over the deuteron magnetic substates gives the adiabatic folding potential,

$$U^{\mathrm{fold}}_{dA}(\mathbf{r}_{dA}) = \int d\boldsymbol{\zeta}\,|\overline{\phi_d}(\boldsymbol{\zeta})|^2\, V_{bA}(\mathbf{r}_{dA}, \boldsymbol{\zeta}), \tag{92}$$

where $|\overline{\phi_d}|^2 \equiv (2J_d + 1)^{-1}\sum_{M_J}|\phi^{M_J}_d(\boldsymbol{\zeta})|^2$ is spherically symmetric. With this convention the odd-parity linear term averages to zero and the spherical part of the quadratic term contributes $\frac{\langle\zeta^2\rangle_d}{24}\nabla^2 V_\Sigma$, while the traceless rank-2 part of the quadratic term has zero spherically averaged expectation and survives as an off-diagonal operator acting between magnetic substates and between $L = 0, 2$ components of the deuteron.

For the equal-mass deuteron specialization,

$$V_{bA} - U_{dA}^{\rm fold} = \underbrace{\tfrac{1}{2}\boldsymbol{\zeta}\cdot\boldsymbol{\nabla}V_\Delta(\mathbf{r}_{dA})}_{\text{E1 (isovector dipole)}} + \underbrace{\tfrac{1}{8}\hat{Q}^{ab}(\boldsymbol{\zeta})\,\partial_a\partial_b V_\Sigma(\mathbf{r}_{dA})}_{\text{E2 (tensor quadrupole)}} + \underbrace{\tfrac{1}{24}(\zeta^2 - \langle\zeta^2\rangle_d)\nabla^2 V_\Sigma(\mathbf{r}_{dA})}_{\text{monopole breathing}} + O(\zeta^3), \quad (93)$$

where $\hat{Q}^{ab}(\boldsymbol{\zeta}) = \zeta^a\zeta^b - \frac{1}{3}\zeta^2\delta^{ab}$ is the traceless rank-2 tensor of the pair. Its Cartesian indices distinguish it from the target-space projector $Q = \mathbf{1} - P$ of Sec. III.

The selection rules of the three pieces in Eq. (93) parallel those established in Sec. V of Ref. [33] for the two-body deuteron-projectile case and are recalled briefly here: the $E1$ piece connects $|\phi_d\rangle$ only to odd-parity ${}^3P$ continuum states because $\langle\phi_d|\boldsymbol{\zeta}|\phi_d\rangle = 0$; the $E2$ piece has zero spherically averaged ground-state expectation by construction of $U_{dA}^{\rm fold}$ [Eq. (92)] but is operator-active through the deuteron $L = 0$–$L = 2$ mixing and on the continuum via the ${}^3S_1$–${}^3D_1$ coupling; the monopole breathing piece vanishes identically on $|\phi_d\rangle$ and enters only through continuum pair states. With surface estimates $|\boldsymbol{\nabla}V_\Delta| \sim 20$ MeV/fm and $|\nabla^2 V_\Sigma| \sim 10$ MeV/fm^2 on ^{208}Pb (from a Lane isovector $|V_p - V_n|/2 \sim$ 5–8 MeV dropping over surface diffuseness $\sim$0.7 fm and a Woods-Saxon isoscalar field with $V_0 \sim 50$ MeV, $R \sim 7$ fm, $a \sim 0.65$ fm), and the deuteron $\zeta_{\rm rms} \equiv (\langle\zeta^2\rangle_d)^{1/2} \approx 3.9$ fm, the three operator pieces of Eq. (93) carry the order-of-magnitude scales

$$\begin{aligned} \tfrac{1}{2}\zeta_{\rm rms}|\boldsymbol{\nabla}V_\Delta|/\sqrt{3} &\sim 20\text{–}25\ \text{MeV}, \\ \tfrac{1}{8}\zeta_{\rm rms}^2|\nabla^2 V_\Sigma| &\sim 15\text{–}25\ \text{MeV}, \\ \tfrac{1}{24}\zeta_{\rm rms}^2|\nabla^2 V_\Sigma| &\sim 5\text{–}8\ \text{MeV}. \end{aligned} \quad (94)$$

These reproduce the estimates of Ref. [33], since the pair moments and target-field inputs are the same.

The overall sign of the $E1$ piece reflects the convention $\boldsymbol{\zeta} = \mathbf{r}_n - \mathbf{r}_p$ adopted here, opposite to Ref. [33]; only $|\boldsymbol{\nabla}V_\Delta|$ enters the scale estimate.

The scales in Eq. (94) do not estimate cross-section corrections. For $L_{\rm diff} \sim 0.6$ fm, $\zeta_{\rm rms}/L_{\rm diff} \sim 6$; higher powers of $\boldsymbol{\zeta}$ are not suppressed. A calculation must therefore use the full coordinate-dependent interaction. The source integral [Eq. (90)] also contains radial phase-space weights and cancellations between distorted waves, which cannot be inferred from surface operator magnitudes. The multipole expansion identifies the selection rules, but its truncated terms do not rank the contributions to the cross section.

In the usual two-body IAV calculation of ${}^6\mathrm{Li} + A \to d + (\alpha + A)^*$, the entrance state is an $\alpha + d$ cluster wave function. The three-body source instead projects Φ_a onto the detected deuteron after the interaction acts. It samples the wave function in the reaction region, so the norm of the deuteron component alone cannot determine the correction. To evaluate this difference, expand Φ_a in pair eigenstates at fixed $\mathbf{y}$:

$$\Phi_a(\boldsymbol{\zeta},\mathbf{y}) = \sum_\nu a_\nu(\mathbf{y})\,\phi_\nu(\boldsymbol{\zeta}), \quad a_\nu(\mathbf{y}) = \langle\phi_\nu|\Phi_a\rangle_\zeta, \quad (95)$$

where ν labels the np pair eigenstate and $\sum_\nu$ includes a discrete sum over bound pair states [only the deuteron for $b = (np)$] and a continuum integral over np scattering states. Substituting into Eq. (90) and integrating over $\boldsymbol{\zeta}$ gives

$$\begin{aligned} &\tilde{\rho}_d^{(\rm ref)}(\mathbf{r}_\alpha) \\ &= \underbrace{\int d\mathbf{r}_b\ \chi_d^{(-)*}(\mathbf{r}_b)\,[U_{dA}(\mathbf{r}_{dA}) - U_d(\mathbf{r}_b)]\,\chi_a^{(+)}(\mathbf{R})\,f_d(\mathbf{y})}_{\boldsymbol{\zeta}\text{-independent operator: pair projection selects } \nu=d} \\ &\quad + \underbrace{\sum_\nu \int d\mathbf{r}_b\ \chi_d^{(-)*}(\mathbf{r}_b)\,\mathcal{F}_{d\nu}(\mathbf{y})\,\chi_a^{(+)}(\mathbf{R})\,a_\nu(\mathbf{y})}_{\text{pair form factors generated by } V_{bk}}, \end{aligned} \quad (96)$$

where $\mathbf{R}$ and $\mathbf{y}$ retain the arguments of Eq. (9). The pair form factor

$$\mathcal{F}_{d\nu}(\mathbf{y}) = \int d\boldsymbol{\zeta}\,\phi_d^*(\boldsymbol{\zeta})\,[V_{n\alpha} + V_{p\alpha}](\mathbf{y},\boldsymbol{\zeta})\,\phi_\nu(\boldsymbol{\zeta}) \quad (97)$$

couples the pair state ϕ_ν to the deuteron through $V_{bk} = V_{n\alpha} + V_{p\alpha}$. For $m_n \approx m_p$, the α-nucleon separations are $\mathbf{y} - \boldsymbol{\zeta}/2$ and $\mathbf{y} + \boldsymbol{\zeta}/2$. Expansion about $\mathbf{y}$ gives

$$\begin{aligned} \mathcal{F}_{d\nu}(\mathbf{y}) &= \langle\phi_d|V_{n\alpha} + V_{p\alpha}|\phi_\nu\rangle_\zeta \\ &= [V_{n\alpha}(\mathbf{y}) + V_{p\alpha}(\mathbf{y})]\,\delta_{d\nu} \\ &\quad - \tfrac{1}{2}[\boldsymbol{\nabla}V_{n\alpha} - \boldsymbol{\nabla}V_{p\alpha}]\cdot\langle\phi_d|\boldsymbol{\zeta}|\phi_\nu\rangle_\zeta \\ &\quad + \tfrac{1}{8}\partial_a\partial_b[V_{n\alpha} + V_{p\alpha}]\langle\phi_d|\zeta^a\zeta^b|\phi_\nu\rangle_\zeta + O(\zeta^3). \end{aligned} \quad (98)$$

The zeroth-order term survives only for $\nu = d$ and gives the ordinary cluster form factor. The remaining terms involve the same pair moments $\langle\phi_d|\boldsymbol{\zeta}^n|\phi_\nu\rangle_\zeta$ as Eq. (93), with the $E1$, $E2$, and monopole selection rules stated there. Here the radial fields come from the $N - \alpha$ interactions and are evaluated at $\mathbf{y}$, rather than from the target fields at $\mathbf{r}_{dA}$.

The reference-channel source in Eq. (96) contains the deuteron spectroscopic amplitude

$$f_d(\mathbf{y}) = a_d(\mathbf{y}) = \langle\phi_d|\Phi_a\rangle_\zeta = \int d\boldsymbol{\zeta}\,\phi_d^*(\boldsymbol{\zeta})\,\Phi_a(\boldsymbol{\zeta},\mathbf{y}), \quad (99)$$

which multiplies $U_{dA} - U_d$. The other pair components enter through $\mathcal{F}_{d\nu}a_\nu$. The term $\nu = d$ is diagonal; $\nu \neq d$ describes mixing with other pair states.

If Φ_a factorizes as $\Phi_a(\boldsymbol{\zeta},\mathbf{y}) = \phi_d(\boldsymbol{\zeta})\,f(\mathbf{y})$ with the deuteron in its ground state, then $a_\nu(\mathbf{y}) = \langle\phi_\nu|\phi_d\rangle_\zeta f(\mathbf{y}) = \delta_{\nu d}\,f(\mathbf{y})$ by the orthogonality of the pair eigenstates, so that $a_d(\mathbf{y}) = f(\mathbf{y})$ and $a_\nu(\mathbf{y}) = 0$ for all $\nu \neq d$. The second line of Eq. (96) then collapses to its single term $\mathcal{F}_{dd}(\mathbf{y})\,f(\mathbf{y})$, and $\mathcal{F}_{dd}$ reduces to the folding-potential correction $\langle\phi_d|V_{n\alpha} + V_{p\alpha}|\phi_d\rangle_\zeta$ that combines with $U_{dA} - U_d$ into the ordinary two-body IAV residual $V_{bx} + U_{bA} - U_b$ of Ref. [33], with the identifications $b \to d$ and $x \to \alpha$ in the two-body framework. In the full three-body wave function, $a_\nu(\mathbf{y}) \neq 0$ for $\nu \neq d$, and the second line of Eq. (96) contributes through the off-diagonal form factors $\mathcal{F}_{d\nu}$; the same structure appears in the explicit coupling source, with $V_{bA} - U_{bA}$ replacing V_{bk} in $\mathcal{F}_{d\nu}$. Constructing the cluster source with the same projected amplitude $f_d = a_d$ isolates this mixing. The difference between the full and cluster

sources is

$$\tilde{\rho}_d^{(\mathrm{ref})} - \tilde{\rho}_d^{(\mathrm{ref,cluster})} = \sum_{v\neq d} \int d\mathbf{r}_b \, \chi_d^{(-)*} \mathcal{F}_{dv}(\mathbf{y}) \, \chi_a^{(+)} \, a_v(\mathbf{y}), \tag{100}$$

so both the nondeuteron amplitudes a_v and the matrix elements of $V_{n\alpha} + V_{p\alpha}$ enter the correction. The leading dipole term couples the even-parity deuteron to odd-parity continuum states, while the quadrupole term couples to even-parity components. The same pair moments enter the explicit pair-target coupling, with different radial fields.

The three-body wave function Φ_a for ^{6}Li can be obtained from established three-body methods, including the Gaussian expansion method [35], the Lagrange-mesh method [36,37], and coupled-channel approaches [38]. The deuteron spectroscopic factor $S_d = \int |f_d(\mathbf{y})|^2 \, d\mathbf{y}$ measures the norm of the $\alpha + d$ component. It is model-dependent and is not an observable. Nor does $1 - S_d$ give the reaction-amplitude correction: Eq. (100) weights the nondeuteron components by the form factors and distorted waves at each radius. These off-diagonal contributions are absent from the two-body IAV cluster model.

Choosing $\alpha = 0$ gives the inclusive deuteron spectrum. Choosing a continuum state $\alpha = (\mathbf{q})$ gives the np coincidence distribution for $^6\mathrm{Li} + A \to (np)_{\mathbf{q}} + (\alpha + A)^*$ at specified pair relative energy and angle. Such a measurement would probe np correlations during breakup.

The same $^6\mathrm{Li} = \alpha + n + p$ system also illustrates Partition B. Relabeling the constituents so that the detected particle is $i = p$, with $j = n$ and $k = \alpha$, the unresolved system is $n + \alpha + A$ and the propagator $G_{jkA} = G_{n\alpha,A}$ describes their three-body propagation. Under this relabeling, the Partition-B Jacobi coordinates defined by Eqs. (4) and (5) become $\boldsymbol{\zeta}' = \mathbf{r}_n - \mathbf{r}_\alpha$ (the $n - \alpha$ relative coordinate, internal to the unresolved pair) and, by the Partition-B relation below Eq. (9), $\mathbf{y}' = \mathbf{r}_{pA} - \mathbf{r}_{(n\alpha)A}$ (the position of the detected proton relative to the $n - \alpha$ center of mass), where $\mathbf{r}_{pA}$ is the exact proton-target separation, $\mathbf{r}_{(n\alpha)A} = (m_n\mathbf{r}_n + m_\alpha\mathbf{r}_\alpha)/(m_n + m_\alpha)$, and $\mathbf{r}_n$, $\mathbf{r}_\alpha$ are the n–A and α–A relative coordinates of the unresolved subsystem, following the convention of Sec. IV. These differ from the Partition-A coordinates $\boldsymbol{\zeta} = \mathbf{r}_n - \mathbf{r}_p$ and $\mathbf{y} = \mathbf{r}_\alpha - (m_n\mathbf{r}_n + m_p\mathbf{r}_p)/(m_n + m_p)$ used earlier in this section; the two parametrizations of the same Φ_a are related by a Jacobi transformation. This observable cannot be reduced to the standard two-body IAV, because the remaining pair $(n\alpha) = {}^5$He is unbound and does not form a two-body cluster. In the CFH reference limit ($V_{pA} \to U_{pA}$), the source, after projection onto the target ground state Φ_A as in Sec. IV, is

$$\tilde{\rho}_p^{(\mathrm{ref})}(\mathbf{r}_n, \mathbf{r}_\alpha) = \int d\mathbf{r}_p \, \chi_p^{(-)*}(\mathbf{r}_p) \times (V_{pn} + V_{p\alpha} + U_{pA} - U_p) \, \chi_a^{(+)} \, \Phi_a(\boldsymbol{\zeta}', \mathbf{y}'), \tag{101}$$

where the integration variable $\mathbf{r}_p$ is the proton Jacobi coordinate, namely, the proton position relative to the center of mass of the unresolved $n + \alpha + A$ system, U_{pA} is evaluated at the exact proton-target separation $\mathbf{r}_{pA} = \mathbf{r}_p + \frac{m_n+m_\alpha}{m_n+m_\alpha+m_A} \mathbf{r}_{(n\alpha)A}$ [Eq. (8)], built from the source output variables, and the projectile coordinates $(\boldsymbol{\zeta}', \mathbf{y}')$ and the entrance argument of $\chi_a^{(+)}$ [Eqs. (4), (5), and (9) with $i = p$, $j = n$, $k = \alpha$] are determined by the exact kinematic relations of Sec. II. After the CFH-type Feshbach-Faddeev reduction of the three-body resolvent, the NEB cross section takes the schematic form $W_n + W_\alpha + W_{3B}$ [29], where W_n describes neutron absorption by the target (forming compound nucleus $A + n$), W_α describes α absorption (forming $A + \alpha$), and W_{3B} represents the correlated $n\alpha$ absorption (forming $A + n + \alpha$). The cross term W_{3B} of CFH [29] has no two-body IAV counterpart. The explicit proton-target coupling adds $V_{pA} - U_{pA}$ to the source. For a structureless proton, this correction represents target excitations driven by the difference between the microscopic $p - A$ interaction and the elastic optical potential. Unlike the deuteron-target coupling of the pair-detection channel, $V_{dA} - U_{dA}$, it has no finite-size tidal enhancement from an extended detected fragment, so its quantitative importance must be assessed through the reaction calculation rather than inferred from the operator scale alone.

For $^9\mathrm{Be} = \alpha + \alpha + n$, the reaction $^9\mathrm{Be} + A \to {}^8\mathrm{Be} + (n + A)^*$ requires an unbound pair state $b = {}^8\mathrm{Be} = (\alpha\alpha)$. Choosing ϕ_α as a continuum or discretized-bin state selects a pair energy interval or resonance window. The source then retains the three-body projectile wave function, whereas Ref. [27] treats ^{8}Be as a preformed pair.

VIII. DISCUSSION

The sum rules and their optical reductions require different assumptions. The conditions are collected here as (A1)–(A6), followed by the numerical steps needed for each measurement. Conditions (A1)–(A3) define the DWBA sum rules and exact rearrangements within them; (A4) specifies the optical representation, (A5) its absorption-sign condition, and (A6) the diagonal approximation for direct propagation between excited target states. The diagonal approximation in (A6) does not require the sign condition in (A5).

(A1) DWBA truncation of the full T-matrix. The inclusive cross section is computed from the DWBA transition amplitudes (14) and (43); residual distortion in the entrance and exit channels is encoded in the optical potentials U_a, U_b, U_i. This is the standard sum-rule starting point and fixes Eqs. (18) and (45).

(A2) Spectral identity for the unresolved subsystem. The on-shell sum over $|\Psi^c\rangle$ is replaced by $-\pi^{-1}\,\mathrm{Im}\,G$ via Sokhotski-Plemelj. This is exact within (A1) and applies to both partitions.

(A3) Feshbach P-Q rewriting in target Hilbert space. The unresolved-system propagator is split as $G = G_{PP} + G_{PQ} + G_{QP} + G_{QQ}^{\mathrm{full}}$ with $P = |\Phi_A\rangle\langle\Phi_A|$, $Q = \mathbf{1} - P$. This is an exact algebraic rearrangement and introduces no approximation. The reduced Q-space resolvent entering the Feshbach kernel is denoted g_Q.

(A4) Single-channel optical reduction in the P sector. The exact Feshbach polarization kernel from G_{PP}^{exact} is replaced by an energy-independent phenomenological optical representation in both partitions. For Partition A this amounts to writing the elastic block of G_{kA}^{full} as the two-body optical

resolvent $G_k = (E^+_{k,\alpha} - K_k - U_k)^{-1}$, with dynamic polarization absorbed into the phenomenological U_k [underlying Eq. (32)]. For Partition B it amounts to replacing $\Delta\mathcal{H}_{PP}$ in Eq. (49) by the CFH-optical kernel $U_j + U_k + \mathcal{V}_{3B}$ of Eq. (56), absorbing the diagonal one-fragment Feshbach pieces into phenomenological optical potentials and retaining the irreducible three-body cross term as $\mathcal{V}_{3B}$ [underlying Eq. (59)]. In both cases, the E derivative of the corresponding exact polarization kernel is neglected; only the genuine three-body cross term $\mathcal{V}_{3B}$ has no Partition-A counterpart.

(A5) Phenomenological absorption-sign assumption. The full kernel $W_j + W_k + W_{3B}$ in Eq. (58) is treated as negative semidefinite under the standard CFH-optical replacement. At the microscopic level this is the negative semidefiniteness of $\mathrm{Im}\,\Delta\mathcal{H}_{PP}$; at the phenomenological level it requires the residual three-body W_{3B} to be a small perturbation around absorptive $W_j + W_k$, an assumption inherited from the standard CFH optical reduction [29] and not introduced here.

(A6) Diagonal-intermediate-target-states ansatz. Only the outer propagation between target-excited labels in the direct g_Q term is diagonalized: $\langle\Phi_{A''}|g_Q|\Phi_{A'}\rangle$ with $A' \neq A''$ is dropped, giving Eq. (66). The nested dressing inside each diagonal block, defined through Eq. (65), is retained and continues to generate the target-excited optical pieces $\tilde{U}^{(A')}_{j,k}$ and $\tilde{\mathcal{V}}^{(A')}_{3B}$. The resulting target-excited CFH-like kernels follow from Eqs. (64)–(68). This is used *only* for the direct g_Q part of the pure-Q contribution $\mathcal{I}_Q$ of Eq. (63); the PQ and QP interferences and the induced $Q \to P \to Q$ part of G^{full}_{QQ} are not reduced under this ansatz and remain in their unreduced Feshbach form.

The optical assumptions (A4) and (A5) concern the reduced absorption formulas, while (A6) concerns the direct target-excited contribution. None of these reductions changes the original DWBA sum rules or their Feshbach decomposition. Relaxing (A4) requires returning to the energy-dependent polarization kernels. The reference interactions U_{bA} and U_{iA} are added and subtracted in the source and cancel before reduction. The separated terms depend on the reference potentials. Within a fixed reduction, only their sum gives the inclusive observable.

For pair detection, the two-body optical propagator can be evaluated by the methods already used in IAV calculations. The extra work lies in the source: Eq. (90) integrates over both the pair center-of-mass coordinate $\mathbf{r}_b$ and the internal coordinate $\boldsymbol{\zeta}$, rather than over $\mathbf{r}_b$ alone. The interactions depend on the individual constituent positions, so the six-dimensional integral does not factorize. It requires a three-body projectile wave function and, for an extended pair such as the deuteron ($\zeta_{\mathrm{rms}} \approx 3.9$ fm), a converged angular-momentum expansion. In the reduced channels where post-prior equivalence has been established, the prior form avoids the slow convergence caused by the lack of a natural cutoff in the post-form transition integral.

For ^{6}Li, a useful comparison begins with the ordinary $\alpha + d$ cluster model, $\Phi_a \approx \phi_d f$, and $V_{bA} \to U_{bA}$. Replacing the factorized wave function by the full Φ_a in Eq. (19) tests projectile correlations. Adding $V_{bA} - U_{bA}$ then tests the explicit pair-target coupling. The two steps separate the effects of projectile structure and target coupling in the calculated spectrum.

For single-particle detection, the CFH reduction [29] requires a three-body wave function acted on by $W_j + W_k + W_{3B}$. The source also contains $V_{iA} - U_{iA}$. For structureless i, this target-excitation coupling can be evaluated with the inelastic form factors used in (p, p') and (α, α') analyses [Eq. (84)]; for composite i, the tidal multipoles of Ref. [33] also enter. Their propagation is treated by the target-space reduction described above.

The diagonal-target approximation also gives a practical form of the three-body cross term in Eq. (54):

$$\mathcal{V}_{3B} \approx \sum_{A' \neq 0} \left[F^{(j)}_{0A'}(\mathbf{r}_j)\, \tilde{G}^{(A')}_{jk}\, F^{(k)}_{A'0}(\mathbf{r}_k) + F^{(k)}_{0A'}(\mathbf{r}_k)\, \tilde{G}^{(A')}_{jk}\, F^{(j)}_{A'0}(\mathbf{r}_j) \right], \tag{102}$$

with $F^{(j)}_{A'0}(\mathbf{r}_j) = \langle\Phi_{A'}|V_{jA}|\Phi_A\rangle$ and $F^{(j)}_{0A'} = [F^{(j)}_{A'0}]^*$ the inelastic transition form factors of $j + A$, and likewise for k. These are the usual $j + A$ and $k + A$ inelastic form factors, obtained from rotational or vibrational couplings for a collective target [38]. At lowest order, $\tilde{G}^{(A')}_{jk}$ is evaluated at $E_{jkA} - \omega_{A'}$ with the Hamiltonian $h_j + h_k + K_j + K_k + V_{jk} + U_j + U_k$. The result is a nonlocal, energy-dependent second-order kernel. A first calculation can set $\mathcal{V}_{3B} = 0$ in the wave equation and evaluate $\langle\psi_{jk}|W_{3B}|\psi_{jk}\rangle$ perturbatively. Its size, and the effect of the off-diagonal target blocks omitted in (A6), must be tested on the resulting wave function.

Only the action of G^{opt}_{PP} on the source is needed. With $|\psi_{jk}\rangle = G^{\mathrm{opt}}_{PP}|\hat{\rho}_{jk}\rangle$, Eq. (59) requires the solution of $(E^+_{jkA} - \mathcal{H}^{\mathrm{opt}}_{PP})|\psi_{jk}\rangle = |\hat{\rho}_{jk}\rangle$ with outgoing boundary conditions, rather than construction of the full off-shell propagator. Expanding the jk pair in bound and continuum-bin states gives inhomogeneous coupled equations that can be solved by the Lagrange-mesh R-matrix method [39,40]. This route has the cost and bin asymptotics of a four-body continuum-discretized coupled channels (CDCC) calculation [28]. Alternatively, a Faddeev-Alt-Grassberger-Sandhas (AGS) treatment provides exact three-body asymptotics and can include complex, nonlocal fragment-target optical potentials [23,41]. In either route, $\mathcal{V}_{3B}$ can first be added perturbatively. The homogeneous equation gives $|\chi^{(+)}_{jk}\rangle$ in Eq. (60).

A first application is $^6\mathrm{Li} + A \to p + (n + \alpha + A)^*$ above the Coulomb barrier. It requires a three-body projectile wave function, the nA, αA, and $n\alpha$ interactions, a six-dimensional prior-form source, and a discretized $n\alpha$ continuum [35,36]. With $\mathcal{V}_{3B} = 0$, each proton energy and angle requires one inhomogeneous solve of four-body-CDCC size.

The normal-eigenstate target reduction does not describe pairing-collective enhancement of two-neutron absorption on an open-shell target. For an unresolved neutron pair in (t, p) or Borromean-projectile reactions, the model-space sum in Eq. (45) includes bound $nn + A$ configurations, but these are not antisymmetrized physical $A + 2$ eigenstates. The subsequent reduction is narrower still: its Q space contains particle-number-conserving states of H_A, whereas pairing vibrations are coherent quasiparticle configurations in the $A \pm 2$ systems [29,42]. That coherence is supplied neither by the

free V_{jk} nor by the diagonal form of Eq. (102). A treatment of this physics requires a quasiparticle target space or explicit two-nucleon transfer channels. The (t, n) reaction leaves an unresolved $p + n$ pair and raises the analogous question for proton-neutron correlations.

In the two-body-projectile limit [$a = d$, $a + A \to p + (n + A)^*$] Partition B (single-particle detection of p, with the unresolved subsystem reducing to $n + A$) reduces to the standard two-body IAV sum rule with a structureless detected particle; this is the same inclusive observable that Ref. [23] solves exactly within the Faddeev framework. Partition A in the same limit corresponds to detecting the bound deuteron itself, which is elastic scattering of d off A rather than an inclusive breakup observable, and is not the relevant two-body limit for the comparison between IAV and Faddeev. Reference [23] is therefore the relevant benchmark for the two-body limit of Partition B.

For ^{6}Li, the deuteron and proton spectra should be calculated with the same projectile wave function and consistent channel interactions and source terms. Proton detection leaves the unbound $n\alpha$ pair and requires the three-body $n + \alpha + A$ propagation. The inclusive α spectrum, by comparison, can also be treated in the two-body $\alpha + d$ cluster model with an unresolved $d + A$ system [14,16].

For the reference pair-target interaction, the adiabatic folding potential [43,44] removes the diagonal correction and leaves off-diagonal pair and target-excitation couplings in the explicit source. An empirical elastic optical potential is also admissible, provided polarization effects already included in it are not counted again [33,45].

IX. SUMMARY AND OUTLOOK

The detected particles determine the unresolved propagator in inclusive breakup of a three-body projectile. Pair detection leaves a two-body propagator and places the projectile's three-body structure in the pair-projected source. Comparing this source with its factorized limit measures, at the amplitude level, what the preformed-cluster approximation omits. Factorizing the projectile wave function recovers the two-fragment detected-cluster result [33].

Single-particle detection requires a three-body propagator. Its optical reduction in the target ground state recovers the CFH kernel $W_j + W_k + W_{3B}$ [29], with the absorption-sign condition (A5). The detected particle's target interaction adds target-excited contributions. The direct part reduces to analogous optical kernels under (A6); off-diagonal propagation and interference remain in Feshbach form. These restrictions belong to the reduced formulas, not to the original DWBA sum rules.

A first calculation can compare deuteron and proton spectra from $^6\mathrm{Li} = \alpha + n + p$ using the same projectile wave function. The deuteron channel requires the pair-projected source; the proton channel requires an inhomogeneous three-body solve. Inelastic form factors provide a starting point for including $\mathcal{V}_{3B}$ perturbatively. Describing pairing-collective enhancement of two-neutron absorption on open-shell targets, relevant to (t, p) and Borromean projectiles, requires a quasiparticle target space or explicit two-nucleon transfer channels beyond the target reduction used here.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this paper. Requests for further information or data should be sent to the author.

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