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Chapter 12

Elements of Formal Scattering Theory

12.1   Scattering States

In the previous chapter, we more or less phenomenologically derived the boundary conditions for the scattering states. We now want to introduce a more formal definition. The time-dependent Schrödinger equation

$$\hbar i \ | \dot{\psi}(t) \rangle \ = \ H \ | \psi(t) \rangle \tag{12.1}$$

with $H = H_0 + V$, and $H_0 = \frac{P^2}{2m}$ has the solution

$$| \psi(t) \rangle \ = \ e^{-\frac{i}{\hbar} \ Ht} \ | \psi(0) \rangle \ . \tag{12.2}$$

A reasonable requirement that (12.2) is a scattering state will be to require that long before and after the scattering process $| \psi(t) \rangle$ behaves like a free wave packet $| \varphi_a(t) \rangle$. Mathematically we can formulate this with the norm as

$$\| \ | \psi_a^{(+)}(t) \rangle - | \varphi_a(t) \rangle \ \| = \| \ e^{-\frac{i}{\hbar} \ Ht} \ | \psi_a^{(+)}(0) \rangle - e^{-\frac{i}{\hbar} \ H_0 t} \ | \varphi_a(0) \rangle \ \| \overset{t \to -\infty}{\longrightarrow} \ 0 \tag{12.3}$$

where the limit $t \to -\infty$ implies that long before the scattering $| \psi_a^{(+)}(0) \rangle$ should behave as a free wave. The index a denotes an arbitrary initial distribution of momenta in the wave packet.

$$\| \ e^{\frac{i}{\hbar} \ Ht} \left(e^{\frac{-i}{\hbar} \ Ht} \ | \varphi_a^{(+)}(0) \rangle \ - \ e^{-\frac{i}{\hbar} \ H_0 t} \ | \varphi_a(0) \rangle \right) \|$$

$$= \ \| \ | \psi_a^{(+)}(0) \rangle - e^{\frac{i}{\hbar} Ht} \ e^{-\frac{i}{\hbar} \ H_0 t} \ | \varphi_a(0) \rangle \ \| \overset{t \to -\infty}{\longrightarrow} \ 0 \ . \tag{12.4}$$

Thus, we can define a scattering state as

$$| \psi_a^{(+)} \rangle \;=\; s - \lim_{t \to -\infty} e^{\frac{i}{\hbar} Ht} \, e^{-\frac{i}{\hbar} H_0 t} \, | \varphi_a \rangle \;=\; \Omega^{(+)} \, | \varphi_a \rangle \tag{12.5}$$

where we defined the Møller operator

$$\Omega^{(+)} \;:=\; s - \lim_{t \to -\infty} W(t) \;=\; s - \lim_{t \to -\infty} e^{\frac{i}{\hbar} Ht} \, e^{-\frac{i}{\hbar} H_0 t} \;. \tag{12.6}$$

In order to show the existence of the Møller operator, we have to show the convergence. Let $t_2 < t_1 < 0$ and consider

$$
\begin{aligned}
\| \, [W(t_2) - W(t_1)] \, \varphi_a \, \| \;&=\; \| \int_{t_1}^{t_2} dt \, \frac{dW(t)}{dt} \, \varphi_a \, \| \\
&\leq\; | \int_{t_1}^{t_2} dt \, \| \, \frac{dW(t)}{dt} \, \varphi_a \, \| \, | \\
&=\; | \int_{t_1}^{t_2} dt \, \| \, e^{\frac{i}{\hbar} Ht} \, \frac{1}{\hbar} \, (H - H_0) \, e^{-\frac{i}{\hbar} H_0 t} \, \varphi_a \, \| \, | \\
&=\; | \int_{t_1}^{t_2} dt \, \frac{1}{\hbar} \, \| \, V \, e^{-\frac{i}{\hbar} H_0 t} \, \varphi_a \, \| \, | \\
&=\; | \int_{t_1}^{t_2} dt \, \frac{1}{\hbar} \, \| \, V \, \varphi_a(t) \, \| \, | \;.
\end{aligned}
\tag{12.7}
$$

The existence of the integral means that the norm $\| \cdot \|$ has to fall off at least as $\frac{1}{|t|^{1+\varepsilon}}$. We already showed that a free wave packet decays as $\frac{1}{|t|^{3/2}}$. If V only exists in a finite range, one only has to integrate over a finite region, and then the fall-off in t is sufficient. To show this, let us consider the norm in (12.7) separately.

$$\| \, V \, \varphi_a(\vec{r}) \, \|^2 \;=\; \int d^3x \, V^2(\vec{x}) \; | \varphi_a(\vec{x}, t) |^2 \tag{12.8}$$

We assume that V is square integrable, i.e., a vector in the Hilbert space. If, e.g., V is local and real, e.g., $e^{-\mu r}/r$, this is certainly fulfilled. (V square integrable means $\int d^3x \, V^2(\vec{x}) < \infty$.) Then (12.8) becomes

$$
\begin{aligned}
\int d^3x \, V^2(\vec{x}) \; | \varphi_a(\vec{x}, t) |^2 \;&\leq\; \int d^3x \; | V(\vec{x}) |^2 \, | \varphi_a(\vec{x}, t) |^2 \\
&\leq\; \int d^3x \, V^2(\vec{x}) \, \frac{c}{t^3} \\
&=\; \frac{c}{t^3} \int d^3x \, V^2(x) \;=\; \frac{c}{t^3} \cdot c'
\end{aligned}
\tag{12.9}
$$

and thus

$$\| \, V \, \varphi_a(\vec{r}) \, \| \; \leq \; \frac{\bar{c}}{|\, t \,|^{3/2}} \; . \tag{12.10}$$

For the original estimate (12.7) then follows

$$\Big| \int_{t_1}^{t_2} dt \; \frac{1}{\hbar} \; \| \, V \, \varphi_a(t) \, \| \; \Big| \; \leq \; \Big| \int_{t_1}^{t_2} dt \; \frac{\bar{c}}{\hbar} \; \frac{1}{|\, t \,|^{3/2}} \; \leq \; \Big| \; \frac{1}{|\, t_2 \,|^{1/2}} \; - \; \frac{1}{|\, t_1 \,|^{1/2}} \; \Big| \; \stackrel{t \to -\infty}{\longrightarrow} \; 0 \tag{12.11}$$

Thus, for square-integrable potentials, the Møller operator exists. If $V \hat{=} V(\vec{r})$, then $V \sim \frac{1}{r^{3/2+\varepsilon}}$ in order to be square integrable. It can actually be shown (Kupsch-Sandhas theorem) that the Møller operators exist if the potential

$$|\, V(\vec{r}) \,| \; \leq \; \frac{c}{r^{1+\varepsilon}} \tag{12.12}$$

for $r \geq R$, i.e., falls off for large r faster than the Coulomb potential. We can always expand a wave packet $|\, \varphi_a \rangle \in \mathcal{H}$ with respect to plane waves

$$|\, \varphi_a \rangle \; = \; \int d^3 p \; |\, \vec{p} \rangle \langle \vec{p} \,|\, \varphi_a \rangle \; = \; \int d^3 p \; |\, \vec{p} \rangle \; \tilde{\varphi}_a(\vec{p}) \; . \tag{12.13}$$

The scattering state is then given by

$$\begin{aligned} |\, \psi^{(+)} \rangle \; &= \; \Omega^{+} \, |\, \varphi_a \rangle \\ &= \; \int d^3 p \; \Omega^{+} \, |\, \vec{p} \,\rangle \; \tilde{\varphi}_a(\vec{p}) \\ &= \; \int d^3 p \; |\, \vec{p} \,\rangle^{(+)} \; \tilde{\varphi}_a(\vec{p}) \end{aligned}$$
$$\tag{12.14}$$

where we define

$$\Omega^{(+)} \; |\, \vec{p} \,\rangle \; = \; |\, \vec{p} \,\rangle^{(+)} \; . \tag{12.15}$$

Here that states $|\, \vec{p} \,\rangle^{(+)}$ correspond to the scattering states. The distribution function $\tilde{\varphi}_a(\vec{p})$ is the same for the free wave packet and the scattering state, only the "basis vectors" change when going to scattering states.

12.2   Properties of the Scattering States

In (12.6) we have defined the Møller operator as limit for $t \to -\infty$ of the operator

$$W(t) \; := \; e^{\frac{i}{\hbar} \, Ht} \; e^{-\frac{i}{\hbar} H_0 t} \; . \tag{12.16}$$

To investigate properties of the Møller operator, we consider

$$\lim_{t\to\infty} \| (W(t) - \Omega^+)\varphi \| = \lim_{t\to\infty} (\| W(t)\varphi \| - \| \Omega^+\varphi \|) = 0 , \tag{12.17}$$

from which follows (since $W(t)$ unitary)

$$\lim_{t\to-\infty} \| W(t)\varphi \| = \| \Omega^{(+)}\varphi \| = \| \varphi \| \tag{12.18}$$

from which we can extract that

$$\langle \varphi \mid \varphi \rangle = \langle \Omega^{(+)}\varphi \mid \Omega^{(+)}\varphi \rangle = \langle \varphi \mid (\Omega^{(+)})^\dagger \, \Omega^{(+)} \mid \varphi \rangle , \tag{12.19}$$

or since $\mid \varphi \rangle$ is an arbitrary state

$$(\Omega^{(+)})^\dagger \, \Omega^{(+)} = \mathbf{1} . \tag{12.20}$$

The Møller operator is an isometric operator (preserves the norm); however, it is **not** a unitary operator, i.e., a left inverse does not exist. Next we consider

$$^{(+)}\langle \vec{p}\,' \mid \vec{p}\, \rangle^{(+)} = \langle \Omega^{(+)}\vec{p}\,' \mid \Omega^{(+)}\vec{p}\, \rangle = \langle \vec{p}\,' \mid (\Omega^{(+)})^\dagger \, \Omega^{(+)} \mid \vec{p}\, \rangle$$
$$= \langle \vec{p}\,' \mid \vec{p}\, \rangle = \delta(\vec{p}\,' - \vec{p}\,) . \tag{12.21}$$

(12.21) shows that the scattering solutions are normalized to a δ-function in the same way the plane waves are. For further studying the properties of the states $\mid p\rangle^{(+)}$, we rewrite $\Omega^{(+)}$ as

$$\Omega^{(+)} = s - \lim_{t\to-\infty} e^{\frac{i}{\hbar} Ht} \, e^{-\frac{i}{\hbar} H_0 t}$$
$$= s - \lim_{t\to-\infty} e^{\frac{i}{\hbar} H(t-\tau)} \, e^{-\frac{i}{\hbar} H_0(t-\tau)}$$
$$= e^{-\frac{i}{\hbar} H\tau} \, \Omega^{(+)} \, e^{\frac{i}{\hbar} H_0\tau} . \tag{12.22}$$

Differentiating with respect to τ and evaluating at $\tau = 0$ leads to

$$0 = \frac{i}{\hbar} H\Omega^{(+)} - \frac{i}{\hbar} \Omega^{(+)} H_0$$

from which the so-called "intertwining relation" results:

$$H\Omega^{(+)} = \Omega^{(+)} H_0 . \tag{12.23}$$

Applying (12.23) on a plane wave state $\mid \vec{p}\rangle$ gives

$$H\Omega^{(+)} \mid \vec{p}\, \rangle = \Omega^{(+)} H_0 \mid \vec{p}\, \rangle$$
$$= \Omega^{(+)} \frac{P^2}{2m} \mid \vec{p}\, \rangle$$
$$= \frac{p^2}{2m} \Omega^{(+)} \mid \vec{p}\, \rangle , \tag{12.24}$$

from which follows

$$H \mid \vec{p} \,\rangle^{(+)} = \frac{p^2}{2m} \mid \vec{p} \,\rangle^{(+)} , \tag{12.25}$$

which states that the full scattering solution $\mid \vec{p} \,\rangle^{(+)}$ multiplied with H gives the same energy as the free solution $\mid \vec{p} \,\rangle$ multiplied with H_0. This also means that the full solution behaves at large distances similar to the free solution, a boundary condition which we just imposed in the previous chapter. We also showed that $\mid \vec{p} \,\rangle^{(+)}$ are eigenstates to the full Hamiltonian H.

Going to the coordinate space representation should give the results of the previous chapter. For the free Schrödinger equation, we have

$$\begin{aligned}
\langle \vec{x} \mid H_0 \mid \vec{p} \,\rangle &= \int d^3x' \, \langle \vec{x} \mid H_0 \mid \vec{x}\,' \rangle \langle \vec{x}\,' \mid \vec{p} \,\rangle \\
&= \int d^3x' \, \langle \vec{x} \mid -\frac{\hbar^2}{2m} \nabla^2 \, \delta(\vec{x} - \vec{x}\,') \mid \vec{x}\,' \rangle \langle \vec{x}\,' \mid \vec{p} \,\rangle \\
&= -\frac{\hbar^2}{2m} \nabla^2 \, \langle \vec{x} \mid \vec{p} \,\rangle = \frac{p^2}{2m} \, \langle \vec{x} \mid \vec{p} \,\rangle
\end{aligned} \tag{12.26}$$

where $\langle \vec{x} \mid \vec{p} \,\rangle = \frac{1}{(2\pi\hbar)^{3/2}} \, exp \left(\frac{i}{\hbar} \, \vec{p} \cdot \vec{x} \right)$. In a similar fashion, we obtain

$$\langle \vec{x} \mid H \mid \vec{p} \,\rangle^{(+)} = \left(-\frac{\hbar^2}{2m} \nabla^2 + V \right) \, \langle \vec{x} \mid \vec{p} \,\rangle^{(+)} = \frac{p^2}{2m} \, \langle \vec{x} \mid \vec{p} \,\rangle^{(+)} \tag{12.27}$$

or with $\langle \vec{x} \mid \vec{p} \,\rangle^{(+)} = \psi_{\vec{p}}^{\dagger}(\vec{x})$

$$\left(-\frac{\hbar^2}{2m} \nabla^2 + V \right) \, \psi_{\vec{p}}^{\dagger}(\vec{x}) = \frac{p^2}{2m} \, \psi_{\vec{p}}^{\dagger}(\vec{x}) , \tag{12.28}$$

which is the stationary Schrödinger equation for positive energies. Again, the eigenstates are not normalizable. However, here this is a consequence; in the considerations of the previous chapter it had to be an assumption. We still will have to show that $\mid \vec{p} \,\rangle^{(+)} = \Omega^{(+)} \mid \vec{p} \,\rangle$ has the correct asymptotic behavior.

12.3 Integral Equation for $\psi_{\vec{p}}^{(+)}(\vec{x})$

Before being able to set up an integral equation, we have to prove the following statement. Given a function $g(t)$, then for $\varepsilon > 0$

$$\lim_{t \to -\infty} g(t) = -\lim_{\varepsilon \to 0} \int_0^{-\infty} dt' \, \varepsilon \, e^{\varepsilon t'} g(t') . \tag{12.29}$$

We start from the right-hand side

$$-\int_0^{-\infty} dt'\; \varepsilon\; e^{\varepsilon t'} g(t') \;=\; -\int_0^{-\infty} \frac{d}{dt'}\left(e^{\varepsilon t'}\right) g(t')\; dt'$$

$$=\; g(0) \;+\; \int_0^{-\infty} dt'\; e^{\varepsilon t'}\; \frac{dg(t')}{dt'}\;.$$

(12.30)

As a remark, the limit and the integral operation can be interchanged if both converge separately. Thus we have

$$\lim_{\varepsilon\to 0}(-\varepsilon)\int_0^{-\infty} dt'\; e^{\varepsilon t'} g(t') \;=\; \lim_{\varepsilon\to 0}\left[g(0) + \int_0^{-\infty} dt'\; e^{\varepsilon t'}\; \frac{dg(t')}{dt'}\right]$$

$$=\; g(0) \;+\; g(-\infty) \;-\; g(0)$$

$$=\; g(-\infty)\;.$$

(12.31)

With this **"Abel Limit"**, we replace the limit with respect to t by a limit $\varepsilon \to 0$. Now we consider the operators

$$\Omega^{(+)} \;=\; W(-\infty) \;=\; -\lim_{\varepsilon\to 0}\int_0^{-\infty} \varepsilon\; e^{\varepsilon t} W(t)\; dt$$

$$=\; -\lim_{\epsilon\to 0}\int_0^{-\infty} \varepsilon\; e^{\varepsilon t}\; e^{\frac{i}{\hbar} Ht}\; e^{-\frac{i}{\hbar} H_0 t}\; dt$$

$$=\; -\lim_{\varepsilon\to 0} \varepsilon \int_0^{-\infty} e^{\frac{i}{\hbar} Ht}\; e^{-\frac{i}{\hbar}(H_0+i\varepsilon)t}\; dt\;.$$

(12.32)

Applying this on a state $|\,\vec{p}\,\rangle$ yields

$$\Omega^{(+)}\,|\,\vec{p}\,\rangle \;=\; -\lim_{\varepsilon\to 0} \varepsilon \int_0^{-\infty} dt\; e^{\frac{i}{\hbar}\,(H-E-i\varepsilon)t}\;|\,\vec{p}\,\rangle$$

$$=\; \lim_{\varepsilon\to 0} \frac{\varepsilon}{\frac{i}{\hbar}\,(H-E-i\varepsilon)}\;|\,\vec{p}\,\rangle$$

$$=\; \lim_{\varepsilon\to 0} i\varepsilon\; \frac{\hbar}{E+i\varepsilon-H}\;|\,\vec{p}\rangle \;=\; \lim_{\varepsilon\to 0} i\varepsilon\; G(z)\;|\,\vec{p}\rangle$$

(12.33)

where in the last step $\hbar \equiv 1$ was set and $z = E + i\varepsilon$. The function

$$G(z) \;=\; \frac{1}{z-H}$$

(12.34)

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is called the **Resolvent of** H. With (12.33) the scattering state can be written as

$$| \vec{p} \, \rangle^{(+)} \; = \; \Omega^{(+)} \; | \, \vec{p} \, \rangle \; = \; \lim_{\varepsilon \to 0} \; i\varepsilon \; G(E + i\varepsilon) \; | \, \vec{p} \rangle \; . \tag{12.35}$$

$G(E + i\varepsilon)$ contains the full Hamiltonian in the denominator and cannot be treated in the form (12.35). Therefore, we want to split to equation in such a way that H_0 and V are in separate pieces. We define the **free Resolvent** as

$$G_0(z) \; = \; \frac{1}{z - H_0} \tag{12.36}$$

and consider

$$G_0^{-1}(z) \; - \; G^{-1}(z) \; = \; (z - H_0) \; - \; (z - H) \; = \; -H_0 \; + \; H \; = \; V \; . \tag{12.37}$$

Multiplying (12.37) from the left with $G_0(z)$ and the right with $G(z)$ gives

$$G_0(z) \, [G_0^{-1}(z) \; - \; G^{-1}(z)] \, G(z)$$
$$= \; G(z) \; - \; G_0(z) \; = \; G_0(z) \, V \, G(z) \; ,$$

from which follows

$$G(z) \; = \; G_0(z) \; + \; G_0(z) \, V \, G(z) \; . \tag{12.38}$$

The relation (12.38) is called **Hilbert identity** or **Second Resolvent equation**, and has already the typical form of an integral equation. Multiplying (12.37) from the right with $G_0(z)$ gives a different form of the Hilbert identity, namely

$$G(z) \; = \; G_0(z) \; + \; G(z) \, V \, G_0(z) \; . \tag{12.39}$$

By comparison of the last terms in (12.38) and (12.39), we find $G_0(z) \, V \, G(z) \; = \; G(z) \, V \, G_0(z)$. Applying (12.38) on a free state $| \, \vec{p} \, \rangle$ gives

$$G(E + i\varepsilon) \; | \, \vec{p} \, \rangle \; = \; G_0(E + i\varepsilon) \; | \, \vec{p} \, \rangle \; + \; G_0(E + i\varepsilon) \, V \, G(E + i\varepsilon) \, | \, p \rangle$$
$$i\varepsilon \, G(E + i\varepsilon) \; | \, \vec{p} \, \rangle \; = \; \frac{i\varepsilon}{E + i\varepsilon - \frac{p^2}{2m}} \; | \, \vec{p} \, \rangle \; + \; i\varepsilon \, G_0(E + i\varepsilon) \, V \, G(E + i\varepsilon) \; | \, \vec{p} \, \rangle \; ,$$

$$\tag{12.40}$$

where $H_0 \, | \, \vec{p} \, \rangle = \frac{p^2}{2m} \; | \, \vec{p} \, \rangle$. Carrying out the lim $\varepsilon \to 0$ gives

$$\lim_{\varepsilon \to 0} \; i\varepsilon \, G(E + i\varepsilon) \; | \, \vec{p} \, \rangle \; = | \, \vec{p} \, \rangle^{(+)} \; = | \, \vec{p} \, \rangle \; + \; G_0(E + i0) \, V \; | \, \vec{p} \, \rangle^{(+)} \; . \tag{12.41}$$

Eq. (12.41) is the **Lippmann-Schwinger equation for states**, and its exists if $V \, | \, \vec{p} \, \rangle^{(+)}$ is a vector out of the Hilbert space, i.e., V is square integrable. The interpretation of (12.41) is that the full scattering state $| \, \vec{p} \, \rangle^{(+)}$ is the sum of an incident plane

wave $|\,\vec{p}\,\rangle$ and a "perturbation" $G_0\,V\,|\,\vec{p}\,\rangle^{(+)}$, which is caused by the potential V. From this we can deduce that (12.41) has the form of the boundary condition as postulated in (11.26). Applying the form (12.39) of the Hilbert identity leads to

$$
\begin{aligned}
|\,\vec{p}\,\rangle^{(+)} \;=\; i\varepsilon\,G(E+i\varepsilon)\;|\,\vec{p}\,\rangle \;&=\; i\varepsilon\,[1+G(E+i\varepsilon)V]\,G_0(E+i\varepsilon)\;|\,\vec{p}\,\rangle \\
&=\; [1+G(E+i\varepsilon)V]\;|\,\vec{p}\,\rangle \\
&=\; \Omega^{(+)}\;|\,\vec{p}\,\rangle\;,
\end{aligned}
\tag{12.42}
$$

from which we deduce a representation of $\Omega^{(+)}$ as

$$
\Omega^{(+)} \;=\; 1+G(E+i\varepsilon)\,V\;.
\tag{12.43}
$$

However, this representation is not very useful since $G(E+i\varepsilon)$ is not known.

12.4   Coordinate Space Representation of the Lippmann-Schwinger Equation

For obtaining a coordinate space representation, we have to form

$$
\begin{aligned}
\langle\vec{x}\,|\,\vec{p}\,\rangle^{(+)} \;&=\; \langle\vec{x}\,|\,\vec{p}\,\rangle \;+\; \int d^3x'\int d^3x''\,\langle\vec{x}\,|\,G_0(E+i0)\;|\,\vec{x}\,'\rangle\langle\vec{x}\,'\,|\,V\,|\,\vec{x}\,''\rangle\langle\vec{x}\,''\,|\,\vec{p}\rangle^{(+)} \\
\psi_{\vec{p}}^{(+)} \;&=\; \frac{1}{(2\pi\hbar)^{3/2}}\,e^{\frac{i}{\hbar}\,\vec{p}\cdot\vec{x}} \;+\; \int d^3x'\int d^3x''\,G_0^{(+)}(\vec{x},\vec{x}\,')\,V(\vec{x}\,',\vec{x}\,'')\,\psi_{\vec{p}}^{(+)}(\vec{x}\,'')\;.
\end{aligned}
\tag{12.44}
$$

If $V(\vec{x}\,',\vec{x}\,'')$ is assumed to be local, i.e., $V(\vec{x}\,',\vec{x}\,'')=V(\vec{x}\,')\,\delta(\vec{x}\,',\vec{x}\,'')$, this reduces to

$$
\psi_{\vec{p}}^{(+)}(\vec{x}) \;=\; \frac{1}{(2\pi\hbar)^{3/2}}\,e^{\frac{i}{\hbar}\,\vec{p}\cdot\vec{x}} \;+\; \int d^3x'\,G_0^{(+)}(\vec{x},\vec{x}\,')\,V(\vec{x}\,')\,\psi_{\vec{p}}^{(+)}(\vec{x}\,')\;.
\tag{12.45}
$$

This is an integral equation, valid for local potentials. The same equation is obtained by solving the stationary Schrödinger equation.

We still have to determine the coordinate space representation of the free Green function

$G_0^{(+)}(\vec{x}, \vec{x}\,')$:

$$
\begin{aligned}
G_0^{(+)} &= \langle \vec{x} \mid G_0(E+i0) \mid \vec{x}\,' \rangle = \langle \vec{x} \mid \frac{1}{E+i0-H_0} \mid \vec{x}\,' \rangle \\
&= \int d^3p'\, \langle \vec{x} \mid \frac{1}{E+i0-H_0} \mid \vec{p}\,\rangle\langle \mid \vec{p}\,' \mid \vec{x}\,' \rangle \\
&= \int d^3p'\, \langle \vec{x} \mid \vec{p}\,'\rangle\langle \vec{p}\,' \mid \vec{x}\,' \rangle \frac{1}{E+i0-\frac{p'^2}{2m}} \\
&= \int d^3p'\, \frac{1}{(2\pi\hbar)^3}\, e^{\frac{i}{\hbar}\,\vec{p}\,'\cdot(\vec{x}-\vec{x}\,')}\, \frac{1}{E+i0-\frac{p'^2}{2m}} \,.
\end{aligned}
$$

(12.46)

Considering that $\vec{p} = \vec{p}\,'$ and using spherical coordinates leads to

$$
\begin{aligned}
G_0^{(+)}(\vec{x}, \vec{x}\,') &= \frac{1}{(2\pi\hbar)^3} \int_0^\infty \int_{-1}^1 \int_0^{2\pi} dp\, p^2\, d\cos\theta\, d\varphi\, e^{\frac{i}{\hbar}\,p(x-x')\cos\theta}\, \frac{1}{E+i0-\frac{p^2}{2m}} \\
&= \frac{m}{2\pi^2\hbar^2} \int_0^\infty dp\, p^2\, \frac{e^{\frac{i}{\hbar}\,p(x-x')} - e^{-\frac{i}{\hbar}\,p(x-x')}}{ip\mid x-x'\mid}\, \frac{1}{2mE+i0-p^2} \\
&= \frac{m}{2\pi^2\hbar^2} \int_{-\infty}^\infty dp\, p\, \frac{1}{2mE+i0-p^2}\, \frac{e^{\frac{i}{\hbar}\,p(x-x')}}{i\mid x-x'\mid} \\
&= \frac{m}{2\pi^2\hbar^2} \int_{-\infty}^\infty dp\, p\, \frac{1}{\frac{2mE}{\hbar^2}+i0-k^2}\, \frac{e^{ip(x-x')}}{i\mid x-x'\mid} \,,
\end{aligned}
$$

(12.47)

where we used $p = \hbar k$ in the last relation. The integrand has poles for positive energies for $\frac{2mE}{\hbar^2} - \hbar^2 k^2 = 0$, and the poles are located at

$$
\hbar k_{1/2} = \pm\sqrt{2mE+i\varepsilon}
$$

(12.48)

from which follows

$$
\begin{aligned}
\hbar k_1 &= +\sqrt{2mE} + i\varepsilon \\
\hbar k_2 &= -\sqrt{2mE} - i\varepsilon \,.
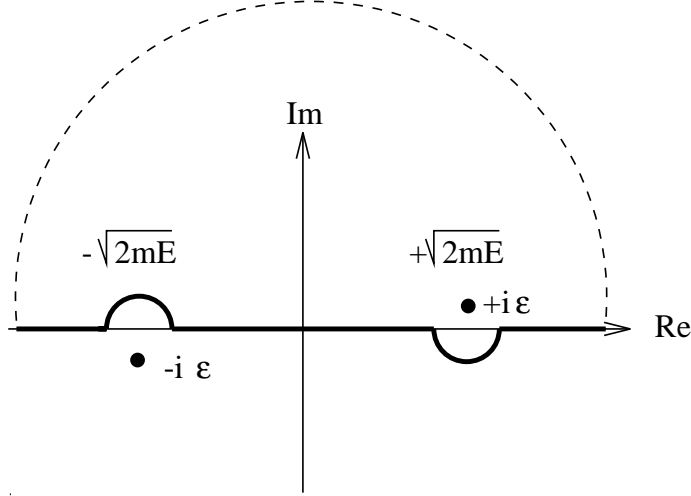\end{aligned}
$$

(12.49)

Fig.

12.1 Integration path for evaluating $G_0^{(+)}(\vec{x}, \vec{x}\,')$.

To solve (12.47) we continue the function into the complex plane and integrate along the path indicated in Fig. 12.1. Since only $\hbar k_1$ lies inside the contour, we obtain for (12.47) along the path of Fig. 12.1

$$\frac{m}{2\pi^2\hbar^3} \quad \int_{-\infty}^{\infty} dk \; \frac{k}{\frac{2mE}{\hbar^2} - k^2} \cdot \frac{e^{ik(x-x')}}{i \mid x - x' \mid} \tag{12.50}$$

$$= \quad \frac{m}{2\pi^2\hbar^3} \; \cdot \; 2\pi i \; Res_{k \to \frac{\sqrt{2mE}}{\hbar}} \; \frac{k}{\frac{2mE}{\hbar^2} - k^2} \frac{e^{ik(x-x')}}{i \mid x - x' \mid} \; .$$

We choose to close the contour in the upper half plane, i.e., have $e^{i(k+i\eta)(x-x')} = e^{ik(x-x')} e^{-\eta(x-x')}$, which falls off sufficiently fast if we take the upper half circle $\to \infty$. The pole for $+\sqrt{2mE}$ sits in the area enclosed by the contour. Applying the Residue theorem leads to the final expression

$$\langle \vec{x} \mid G_0(E + i0) \mid \vec{x}\,' \rangle \; = \; -\frac{m}{2\pi\hbar^2} \frac{e^{i\,\frac{p}{\hbar}\,(x-x')}}{(x - x')} \; . \tag{12.51}$$

With this, (12.45) becomes

$$\psi_{\vec{p}}^{(+)}(\vec{x}) \; = \; \frac{1}{(2\pi\hbar)^{3/2}} \; e^{\frac{i}{\hbar}\,\vec{p}\cdot\vec{x}} \; - \; \frac{m}{2\pi\hbar^2} \int d^3x' \; \frac{e^{\frac{i}{\hbar}\,\vec{p}\cdot(\vec{x}-\vec{x}\,')}}{\mid \vec{x} - \vec{x}\,' \mid} \; V(\vec{x}\,') \; \psi_{\vec{p}}^{(+)}(\vec{x}\,') \tag{12.52}$$

which is the Lippmann-Schwinger equation in coordinate space representation.

12.5 Asymptotic Behavior

In order to make a comparison with the asymptotic boundary conditions imposed in Chapter 11 and better understand their origin, we have to study the asymptotic behavior of (12.52). The first term of the right-hand side corresponds to a plane wave and does already have the desired asymptotic form. We need to study the second term and find out if it behaves asymptotically as an outgoing spherical wave. Expanding $\mid \vec{x} - \vec{x}' \mid^2$ gives

$$
\mid \vec{x} - \vec{x}' \mid^2 = \vec{x}^{\,2} - 2\vec{x} \cdot \vec{x}' + \vec{x}'^{\,2} \approx r^2 - 2\vec{x} \cdot \vec{x}'
$$
$$
= r^2 \left(1 - \frac{2\vec{x} \cdot \vec{x}'}{r^2} \right) \qquad (12.53)
$$

and thus

$$
\mid \vec{x} - \vec{x}' \mid = \sqrt{(\vec{x} - \vec{x}')^2} \approx r \sqrt{\left(1 - \frac{2\vec{x} \cdot \vec{x}'}{r^2} \right)} \approx r \left(1 - \frac{\vec{x}\vec{x}'}{r^2} \right)
$$
$$
= r - \frac{\vec{x} \cdot \vec{x}'}{r} \, ,
$$
$$
(12.54)
$$

which corresponds to a dipole approximation. Inserting (12.54) into (12.52) and considering that

$$
G_0^{(+)}(\vec{x} - \vec{x}') = \frac{m}{2\pi\hbar^2} \frac{e^{ikr}}{r} e^{-i\left(\frac{k\vec{x}}{r} \right) \cdot \vec{x}'} \, ,
$$

we obtain as asymptotic behavior of (12.52)

$$
\psi_{\vec{p}}^{(+)} \overset{r \to \infty}{\longrightarrow} \varphi(\vec{x}) + \left(\frac{-m}{2\pi\hbar^2} \right) \int d^3x' \, e^{-\frac{ik\vec{x}}{r} \cdot \vec{x}'} V(\vec{x}') \, \psi_{\vec{p}}^{(+)}(\vec{x}') \cdot \frac{e^{ikr}}{r} \qquad (12.55)
$$

which, when compared to the "Sommerfeld radiation condition" (11.30), shows that the Lippmann-Schwinger equation has the correct asymptotic behavior. A comparison with (11.30)

$$
\psi_{\vec{p}}^{(sc)}(\vec{x}) \overset{r \to \infty}{\longrightarrow} \varphi_{\vec{p}}(\vec{x}) + f(\hat{p}', \hat{p}) \frac{e^{ikr}}{r} \frac{1}{(2\pi\hbar)^{3/2}}
$$

gives

$$
f(\hat{p}', \hat{p}) = -m \, 4\pi^2\hbar \int d^3\vec{x}' \, \frac{e^{-\frac{i}{\hbar} \vec{p}' \cdot \vec{x}'}}{(2\pi\hbar)^{3/2}} V(\vec{x}') \, \psi_{\vec{p}}^{(+)}(\vec{x}')
$$
$$
= -4\pi^2 m\hbar \int d^3x' \, \langle \vec{p}' \mid \vec{x}' \rangle \langle \vec{x}' \mid V \mid \vec{x}' \rangle \langle \vec{x}' \mid \vec{p} \rangle^{(+)}
$$
$$
= -4\pi^2 m\hbar \, \langle \vec{p}' \mid V \mid \vec{p} \rangle^{(+)} \, .
$$
$$
(12.56)
$$

In order to solve for $f(\hat{p}\,',\hat{p})$, one must first solve the Lippmann-Schwinger equation to obtain $|\,\vec{p}\,\rangle^{(+)}$. Similar to the considerations following (11.91), one can consider solving for $f(\hat{p}\,',\hat{p}\,)$ only in an approximate fashion and expand $|\,\vec{p}\,\rangle^{(+)}\,=|\,p\rangle+\cdots$. If one considers only the first term in the expansion, one obtains the **Born approximation for the scattering amplitude**

$$f(\hat{p}\,',\hat{p}\,)_{Born}\;=\;-4\pi^{2}m\hbar\;\langle\vec{p}\,'\,|\,V\,|\,\vec{p}\,\rangle\;. \tag{12.57}$$

12.6 The Lippmann-Schwinger Equation as Operator Equation

In its general form (12.52) can be written as

$$|\,\psi_{\vec{p}}^{(+)}\rangle\;=\;|\,\varphi_{p}\rangle\;+\;G_{0}^{(+)}\,V\,|\,\psi_{p}^{(+)}\rangle\;. \tag{12.58}$$

Multiplying (12.58) with V and defining $V\,|\,\psi_{p}^{(+)}\rangle\;=\;T\,|\,\varphi_{p}\rangle$ yields

$$T\,|\,\varphi_{p}\rangle\;=\;V\,|\,\varphi_{p}\rangle\;+\;V\,G_{0}^{(+)}\,T\,|\,\varphi_{p}\rangle\;. \tag{12.59}$$

Since the plane waves $|\,\varphi_{p}\rangle$ form a complete set of states, (12.59) is valid as operator equation:

$$T\;=\;V\;+\;V\,G_{0}^{(+)}\,T\;, \tag{12.60}$$

which is the **Operator Lippmann-Schwinger Equation** or **t-matrix** equation. In its momentum space representation (12.60) reads

$$\langle\vec{p}\,'\,|\,T\,|\,\vec{p}\,\rangle\;=\;\langle\vec{p}\,'\,|\,V\,|\,\vec{p}\,\rangle\;+\;\int d^{3}p''\,d^{3}p'''\,\langle\vec{p}\,'\,|\,V\,|\,\vec{p}\,''\rangle\langle\vec{p}\,''\,|\,G_{0}^{(+)}\,|\,\vec{p}\,'''\rangle\langle\vec{p}\,'''\,|\,T\,|\,\vec{p}\,\rangle$$

$$=\;\langle\vec{p}\,'\,|\,V\,|\,\vec{p}\,\rangle\;+\;\int d^{3}p''\,\langle\vec{p}\,'\,|\,V\,|\,\vec{p}\,''\rangle\,\frac{1}{E+i\varepsilon-\frac{p''^{2}}{2m}}\,\langle\vec{p}\,''\,|\,T\,|\,\vec{p}\,\rangle\;. \tag{12.61}$$

This is an integral equation of Fredholm type, and one has to solve for all values of p'', i.e., off-the-energy-shell. However, one needs for the calculation of physical observables only the values for which $\frac{p''^{2}}{2m}\;=\;E\;=\;\frac{p^{2}}{2m}$, i.e., one needs only the on-shell values. The reason for this is that for the calculation of cross sections only the asymptotic expansion was used (compared structure of proofs in Chapter 11). The terminology is as follows: Matrix elements for which

$$p'2\;=\;p^{2}\;=\;p_{0}^{2}\quad\text{are called on}-\text{shell}\;,$$

those for which

$$p'^2 \neq p^2 = p_0^2 \quad \text{are called half} - \text{shell},$$

and those for which

$$p'^2 \neq p^2 \neq p_0^2 \quad \text{are called fully} - \text{off} - \text{shell}.$$

Here p_0^2 is defined via the incoming energy $E_{p_0} = \frac{p_0^2}{2m}$. In terms of the t-matrix, (12.56) can be written as

$$f(\hat{p}', \hat{p}) = -4\pi^2 m\hbar \langle \vec{p}' \mid T \mid \vec{p} \rangle . \tag{12.62}$$

Let us consider the propagator $G_0^{(+)}$ in (12.61). If we write it as

$$\langle \vec{p}' \mid G_0^{(+)} \mid \vec{p} \rangle = \frac{\delta(\vec{p}' - \vec{p})}{E + i\varepsilon - \frac{p'^2}{2m}}$$

$$= \delta(\vec{p}' - \vec{p}) \left[\frac{\mathcal{P}}{E - \frac{p^2}{2m}} - i\pi \, \delta\left(E - \frac{p^2}{2m}\right) \right], \tag{12.63}$$

where we used the **Cauchy Principal value**

$$\frac{1}{x + i\varepsilon} = \frac{\mathcal{P}}{x} - i\pi \, \delta(x) \tag{12.64}$$

then we can consider the approximation

$$\langle \vec{p}' \mid G_0^{(+)} \mid \vec{p} \rangle \approx \delta(\vec{p}' - p')(-i\pi) \, \delta\left(E - \frac{p^2}{2m}\right), \tag{12.65}$$

which is called k-matrix Born approximation.

The Lippmann-Schwinger equation (12.60) is an integral equation of Fredholm type and can, in principal, be solved by iteration (Neumann series) if the kernel $V \, G_0^{(+)}$ is small, so that the convergence of the series is secured. In terms of (12.60), this iteration can be written out as

$$\begin{aligned}
T^{(1)}(z) &= V \\
T^{(2)}(z) &= V + V \, G_0^{(+)} \, V \\
T^{(3)}(z) &= V + V \, G_0^{(+)} \, V \, G_0^{(+)} \, V .
\end{aligned}$$
$$\tag{12.66}$$

In general this series can be written as

$$T(z) \; = \; \sum_{n=1}^{\infty} V(G_0^{(+)}(z)V)^{n-1} \; = \; V \sum_{n=0}^{\infty} (G_0^{(+)}(z)V)^n \qquad (12.67)$$

and is called **Born Series**.

We could have also started from the Lippmann-Schwinger equation for states as given in (12.58). If we define as zeroth order to the full solution $\mid \psi_{\vec{p}}^{(+)} \rangle^0 \; := \; \mid \varphi_{\vec{p}} \rangle$, the free solution, then we obtain in first order iteration

$$\mid \psi_{\vec{p}}^{(+)} \rangle^1 \; = \; \mid \varphi_{\vec{p}} \rangle \; + \; G_0^{(+)} V \mid \varphi_{\vec{p}} \rangle \qquad (12.68)$$

and in general in n-th order

$$\mid \psi_{\vec{p}}^{(+)} \rangle^n \; = \; \sum_{\nu=0}^{n} (G_0^{(+)}V)^{\nu} \mid \varphi_{\vec{p}} \rangle \; . \qquad (12.69)$$

If this series converges for $n \to \infty$, we obtain in this way a solution for (12.58), which reads

$$\mid \psi_{\vec{p}}^{(+)} \rangle \; = \; \sum_{\nu=0}^{\infty} (G_0^{(+)}V)^{\nu} \mid \varphi_{\vec{p}} \rangle \qquad (12.70)$$

and one can say: If the series (12.70) converges, then the vector defined by the series is a proper scattering state. As we have seen in (12.56), the scattering amplitude can be written as

$$f_E(\hat{p}', \hat{p}) \; = \; -4\pi^2 m\hbar \, \langle \varphi_{\vec{p}\,'} \mid V \mid \psi_{\vec{p}}^{(+)} \rangle \; .$$
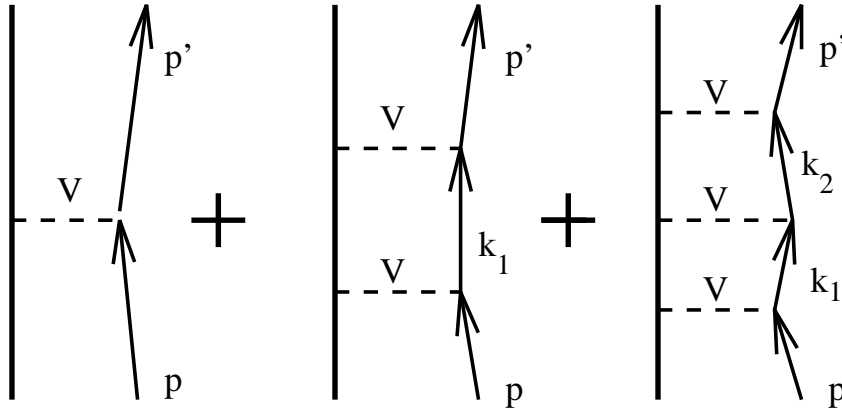
Inserting (12.70) yields

$$f_E(\hat{p}', \hat{p}) \; = \; -4\pi^2 m\hbar \sum_{\nu=0}^{\infty} \langle \varphi_{\vec{p}\,'} \mid V(G_0^{(+)}V)^{\nu} \mid \varphi_{\vec{p}} \rangle \; . \qquad (12.71)$$

Thus, the scattering amplitude can be written as a series

$$f_E(\hat{p}', \hat{p}) \; = \; \sum_{i=1}^{\infty} f_E^{(i)}(\hat{p}', \hat{p})$$
$$f_E^{(i)}(\hat{p}', \hat{p}) \; = \; -4\pi^2 m\hbar \, \langle \varphi_{\vec{p}'} \mid V(G_0^{(+)}V)^{(i-1)} \mid \varphi_{\vec{p}} \rangle \; .$$
$$(12.72)$$

Here (i) counts the power of V. The **first** Born approximation for the scattering amplitude, sometimes called **the** Born approximation, is linear in the potential. It is useful to consider a graphical representation of (12.72).

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Fig. 12.2 Schematic interpretation of the terms in the perturbation series for $f_E(\hat{p}\,',\hat{p})$.

This graphical series expresses the fact that the scattering can be interpreted as a multiple interaction of the potential. The Born approximation is in this respect also called tree approximation. The line $\vec{k}$ in the second-order approximation characterizes $G_0^{(+)}$, i.e., the free motion of the particle between the multiple scatterings. However, this is not the true free motion since in

$$V \; G_0^{(+)}V \;=\; \int \; d^3k_1 \; V \; G_0^{(+)} \mid \vec{k}_1\rangle\langle\vec{k}_1 \mid V \tag{12.73}$$

the momentum $\vec{k}_1$ is an integration variable, and we do not have $\frac{k_1^2}{2m} = \frac{p^2}{2m}$, i.e., $\vec{k}_1$ is not restricted to the on-shell value. We rather have a typical quantum mechanical behavior, during the interaction we have a momentum uncertainty allowed by the uncertainty principle. To stress again, this point of view is only meaningful if the Born series converges.

In the previous considerations, we started from the Lippmann-Schwinger equation (12.58) and derived the Born series (12.70). We also could have started from (12.70) as **definition** of the scattering state, assuming that this definition is only valid within the radius of convergence of the series. Then we can derive the Lippmann-Schwinger equation starting

from (12.70).

$$
\begin{aligned}
\mid \psi_{\vec{p}}^{(+)} \rangle \; &= \; \sum_{\nu=0}^{\infty} (G_0^{(+)}V)^{\nu} \; \mid \varphi_{\vec{p}} \rangle \\
&= \; \mid \varphi_{\vec{p}} \rangle \; + \; G_0^{(+)}V \sum_{\nu=1}^{\infty} (G_0^{(+)}V)^{\nu-1} \; \mid \varphi_{\vec{p}} \rangle \\
&= \; \mid \varphi_{\vec{p}} \rangle \; + \; G_0^{(+)}V \sum_{\nu=0}^{\infty} (G_0^{(+)}V)^{\nu} \; \mid \varphi_{\vec{p}} \rangle \\
&= \; \mid \varphi_{\vec{p}} \rangle \; + \; G_0^{(+)}V \; \mid \psi_{p}^{(+)} \rangle \; .
\end{aligned}
$$

(12.74)

This shows that we "summed up" the Born series. However, we do not obtain an explicit solution on the right-hand side, but rather the unknown function again. This indicates that in general we do not obtain a closed solution for the scattering state but rather an integral equation.

We can interpret this in a slightly different way. As we shall see, the Born series converges for sufficiently high energies. In that region (12.70) is a reasonable definition and within the radius of convergence the Born series and the Lippmann-Schwinger equation are equivalent. This stays valid for all other energies and thus determines the continuation of the definition (12.70) to energy regions, where the series does not exist.

It is appropriate to describe this approach here since it shows how one obtains an integral equation by summing up multiple interactions of V. This procedure is important with respect to quantum field theory. There the usual practice leads to a Dyson series (an infinite series corresponding to the Born series, represented by Feynman diagrams).

Summing up this series gives an integral equation, which resembles in its form the Lippmann-Schwinger equation, the **Bethe-Salpeter equation**, which, however, has a much more complicated structure.

12.7 The Lippmann-Schwinger Equation for the Bound State

If we assume that V supports a bound state $\mid \psi_b \rangle$ at $E = E_b < 0$, then the Schrödinger equation reads

$$
(H_0 + V) \; \mid \psi_b \rangle \; = \; E_b \mid \psi_b \rangle
$$

(12.75)

or

$$(H_0 - E_b) \mid \psi_b \rangle \ = \ -V \mid \psi_b \rangle \ . \tag{12.76}$$

Since $E_b < 0$, there is no regular solution for the case $V = 0$ and we can write

$$\mid \psi_b \rangle \ = \ \frac{1}{E_b - H_0} \ V \mid \psi_b \rangle \ , \tag{12.77}$$

which is the homogeneous Lippmann-Schwinger equation for $\mid \psi_b \rangle$. Evaluating the free Green's function for $E = E_b < 0$ (12.51), we see that $\langle \vec{x} \mid \psi_b \rangle \equiv \psi_b(\vec{x})$ has the correct exponential fall-off behavior, namely proportional to $e^{-\sqrt{2m|E_b|}} \frac{|x-x'|}{|x-x'|}$.

12.8 The Low Equation

When deriving the Lippmann-Schwinger equation for states or operators, we started from the Hilbert identity given by (12.38). Now we want to start from the alternative form (12.39), namely

$$G(z) \ = \ G_0(z) \ + \ G(z) \ V \ G_0(z) \ .$$

Similar to Section 12.3, we apply (12.39) on a free state $\mid \vec{p} \rangle$, multiplying with $i\varepsilon$ and consider the limit $\varepsilon \to 0$.

$$i\varepsilon \ G(E + i\varepsilon) \mid \vec{p} \rangle \ = \ \frac{i\varepsilon}{E + i\varepsilon - \frac{p^2}{2m}} \mid \vec{p} \rangle \ + \ G(E + i\varepsilon) \ V \ \frac{i\varepsilon}{E + i\varepsilon \ \frac{p^2}{2m}} \mid \vec{p} \rangle \ . \tag{12.78}$$

Taking the limit $\varepsilon \to 0$ gives

$$\mid \vec{p} \rangle^{(+)} \ = \ \mid \vec{p} \rangle \ + \ G(E + i\varepsilon) \ V \ \mid \vec{p} \rangle \ , \tag{12.79}$$

which is the **Low equation for states** (corresponding to (12.41)). Multiplying with V and taking into account the definition $V \mid p \rangle^{(+)} = T \mid p \rangle$ gives

$$V \mid \vec{p} \rangle^{(+)} \ = \ T \mid p \rangle \ = \ (V + V \ G \ V) \mid \vec{p} \rangle \tag{12.80}$$

or as operator equation

$$T \ = \ V + V \ G \ V \ . \tag{12.81}$$

For practical calculations, the Low equation is not particularly useful, since it contains the full resolvent $G(z)$. If we assume that the Hamiltonian H has bound states with

$H \mid \psi_b \rangle = E_b \mid \psi_b \rangle$ and a continuous spectrum, then we can write the spectral decomposition of $G(z)$ as

$$\frac{1}{E \pm i\varepsilon - H} = \sum_b \mid \psi_b \rangle \frac{1}{E \pm i\varepsilon - E_b} \langle \psi_b \mid$$
$$+ \int d^3 p \mid \psi_{\vec{p}}^{(+)} \rangle \frac{1}{E \pm i\varepsilon - H} \langle \psi_{\vec{p}}^{(+)} \mid .$$

(12.82)

$G(z)$ is a holomorphic function for $Im \neq 0$, and $G(z)$ has poles at the binding energies E_b. Inserting (12.82) into (12.81) leads to

$$\langle \vec{p}\,' \mid T \mid \vec{p}\, \rangle = \langle \vec{p}\,' \mid V \mid \vec{p}\, \rangle + \sum_b \frac{\langle \vec{p}\,' \mid V \mid \psi_b \rangle \langle \psi_b \mid V \mid \vec{p}\, \rangle}{E - E_b}$$
$$+ \int d^3 k \, \frac{\langle \vec{p}\,' \mid T^\dagger \mid \vec{k}\, \rangle \langle \vec{k}\, \mid T^- \mid \vec{p}\, \rangle}{E \pm i\varepsilon - E_k} ,$$

(12.83)

where the subscript $\pm$ refers to $\pm i\varepsilon$ in $G(E \pm i\varepsilon)$. If $E > 0$, then there are no bound states and the discrete sum over b vanishes. If $E < 0$ is allowed, then $\langle \vec{p}\,' \mid T \mid \vec{p}\, \rangle$ has poles by $E = E_b$, and the residue is separable. [Remark: A function $f(\vec{q}\,', \vec{q}\,)$ is called separable if $f(q', q) = f_1(q')\, f_2(q)$.] The residue can be written as

$$\langle \vec{p}\,' \mid V \mid \psi_b \rangle = \langle \vec{p}\,' \mid (H - H_0) \mid \psi_b \rangle = (E - E_{\vec{p}}\,) \langle \vec{p}\,' \mid \psi_b \rangle$$
$$= (E - E_{\vec{p}}\,)\, \psi_b(\vec{p}\,') ,$$

(12.84)

where $\psi_b(\vec{p}\,')$ is the bound state wave function for $E = E_b$. Thus, if $E < 0$ is close to E_b, then the t-matrix is dominated by the pole term. The residue is characterized by the corresponding bound state wave functions. As function of $E\langle \vec{p}\,' \mid T(E) \mid \vec{p}\rangle$, has the following behavior:
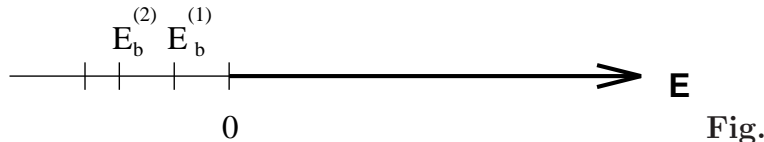
12.3 Pole structure of $\langle \vec{p}\,' \mid T(E) \mid \vec{p}\, \rangle$.

There are poles for the bound states and a cut for the continuous spectrum.

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12.9 Unitarity Relations

When considering only positive energies, the sum over b vanishes in (12.83). Let us consider the difference of $T^{(+)}$ and $T^{(-)}$ as obtained from (12.83) under the assumption that V is hermitian.

$$
\begin{aligned}
\langle \vec{p}' \mid T^{(+)}(E) \quad &- T^{(-)}(E) \mid \vec{p} \rangle = \langle \vec{p}' \mid T^{(+)}(E) \mid \vec{p} \rangle - \langle \vec{p}' \mid T^{(-)}(E) \mid \vec{p} \rangle \\
&= \int d^3 k \langle \vec{p}' \mid T^{(+)}(E) \mid \vec{k} \rangle \left[\frac{1}{E - E_k + i\varepsilon} - \frac{1}{E - E_k - i\varepsilon} \right] \langle \vec{k} \mid T^{(-)}(E) \mid \vec{p} \rangle .
\end{aligned}
\tag{12.85}
$$

Using the Cauchy principal value $\frac{1}{x \pm i\varepsilon} = \frac{\mathcal{P}}{x} \mp i\pi\delta(x)$, we obtain

$$
\begin{aligned}
\langle \vec{p}' \mid T^{(+)}(E) \quad &- T^{(-)}(E) \mid \vec{p} \rangle \tag{12.86} \\
&= -2\pi i \int d^3 k \langle \vec{p}' \mid T^{(+)}(E_k) \mid \vec{k} \rangle \delta(E - E_k) \langle \vec{k} \mid T^{(-)}(E_k) \mid \vec{p} \rangle \\
&= -2\pi i \int_0^\infty dk\, k^2 \int d\Omega_k \langle \vec{p}' \mid T^{(+)}(E_k) \mid \vec{k} \rangle \delta(k'' - k) \frac{m}{k} \langle \vec{k} \mid T^{()}(E_k) \mid \vec{p} \rangle
\end{aligned}
\tag{12.87}
$$

In the last relation we used $\delta(ax) = \frac{1}{|a|}\delta(x)$, $\delta(\varphi(x))] = \sum_i \frac{1}{|\varphi'(a_i)|}\delta(x - a_i)$, where a_i are the roots of the equation $\varphi(x) = 0$, and $E = \frac{p''^2}{2m}$.

Eq. (12.87) is called the **off-shell unitarity relation**, since in general $\mid \vec{p} \mid \neq \mid \vec{p}' \mid \neq \mid \vec{k} \mid$.

For elastic scattering, i.e., $\mid \vec{p} \mid = \mid \vec{p}' \mid$, Eq. (12.87) simplifies to

$$
\begin{aligned}
\langle \vec{p}' \mid T^{(+)}(E_p) \quad &- \quad T^{(-)}(E_p) \mid \vec{p} \rangle \tag{12.88} \\
&= -2\pi i\, m\, p \int d\Omega_k \langle \vec{p}' \mid T^{(+)}(E_p) \mid \vec{k} \rangle \langle \vec{k} \mid T^{(-)}(E_p) \mid \vec{p} \rangle
\end{aligned}
$$

which is usually referred to as **on-shell unitarity relation**.

If we restrict ourselves to the forward direction, i.e., $\vec{p}' = \vec{p}$, or equivalently $\theta = 0$, with θ being the angle between $\vec{p}'$ or $\vec{p}$, we obtain

$$
\begin{aligned}
\langle \vec{p} \mid T^{(+)}(E) - T^{(-)}(E) \mid \vec{p} \rangle &= \langle \vec{p} \mid T^{(+)} \mid \vec{p} \rangle - \langle \vec{p} \mid T^{(+)} \mid \vec{p} \rangle^* \\
&= 2i\, Im\, \langle \vec{p} \mid T^{(+)} \mid \vec{p} \rangle
\end{aligned}
\tag{12.89}
$$

and with (12.89)

$$Im \langle \vec{p} \mid T^{(+)}(E) \mid \vec{p} \rangle = -\pi \, mp \int d\Omega_k \mid \langle \vec{p} \mid T^{(+)}(E) \mid \hat{k} \rangle \mid^2 . \qquad (12.90)$$

Eq. (12.90) is referred to as **optical theorem** and provides a non-linear relation between the imaginary part of T in forward direction and the absolute value of T integrated over all angles.

12.10 S-Operator

In (12.4) the scattering state was defined via its behavior for large negative times, i.e., long before the scattering event.

$$\mid \psi_a^{(+)} \rangle = s - \lim_{t \to -\infty} e^{iHt} e^{-iH_0 t} \mid \varphi_a \rangle = \Omega^{(+)} \mid \varphi_a \rangle .$$

In order to completely characterize the scattering event, we need to have the behavior of $\mid \psi_a^{(+)} \rangle$ for large positive t, i.e.,

$$\mid \psi_a^{(+)}(t) \rangle = e^{-iHt} \mid \psi_a^{(+)}(0) \rangle = e^{-iHt} \Omega^{(+)} \mid \varphi_a \rangle . \qquad (12.91)$$

One of the basic boundary condition was that long after the scattering, the state $\mid \psi_a^{(+)}(t) \rangle$ should again behave as a free wave. Thus, in order to characterize the scattering, we should consider the probability coefficients $\mid \langle \varphi_b(t) \mid \psi_a^{(+)}(t) \rangle \mid^2$ for large times t. Here $\mid \varphi_b(t) \rangle$ is a free wave.

Define

$$\begin{aligned}
S_{ab} &= \lim_{t \to -\infty} \langle \varphi_b(t) \mid \psi_a^{(+)}(t) \rangle \\
&= \lim_{t \to \infty} \langle e^{-iH_0 t} \varphi_b \mid e^{-iHt} \psi_a^{(+)} \rangle \\
&= \lim_{t \to \infty} \langle e^{iHt} e^{-H_0 t} \varphi_b \mid \psi_a^{(+)} \rangle \\
&= \langle \Omega^{(-)} \varphi_b \mid \psi_a^{(+)} \rangle = \langle \psi_b^{(-)} \mid \psi_a^{(+)} \rangle
\end{aligned}$$

$$(12.92)$$

with

$$\Omega^{(-)} := \lim_{t \to \infty} e^{iHt} e^{-iH_0 t} \qquad (12.93)$$

and

$$\Omega^{(-)} \mid \varphi_b \rangle := \mid \psi_b^{(-)} \rangle \qquad (12.94)$$

which characterizes the outgoing scattering state. The existence of $\Omega^{(-)}$ is guaranteed, since the proofs of Section 12.1 never explicitly used the limit $t \to \infty$. Thus, the transition probability from an incoming state to an outgoing state is given by

$$S_{ab}^2 = |\langle \psi_b^{(-)} | \psi_a^{(+)} \rangle|^2 \ . \tag{12.95}$$

With

$$
\begin{aligned}
S_{ab} = \langle \psi_b^{(-)} | \psi_a^{(+)} \rangle &= \langle \Omega^{(-)} \varphi_b | \Omega^{(+)} \varphi_a \rangle \\
&= \langle \varphi_b | \Omega^{(-)\dagger} \Omega^{(+)} | \varphi_a \rangle := \langle \varphi_b | S | \varphi_a \rangle
\end{aligned}
\tag{12.96}
$$

we define the **scattering operator** (S-matrix)

$$S = \Omega^{(-)\dagger} \Omega^{(+)} \ . \tag{12.97}$$

Using the explicit definition of the Møller operators, it can be easily seen that S is unitary. Furthermore, the existence of S depends on the existence of the Møller operators.

12.11   Energy Conservation

In (12.23) the intertwining relation was given as

$$H\Omega^{(\pm)} = \Omega^{(\pm)} H_0 \ .$$

Its complex conjugate equation reads

$$(\Omega^{(\pm)})^\dagger H = H_0 \Omega^{\pm\dagger} \ .$$

From the isometry property of the Møller operators, $\Omega^{(\pm)\dagger} \Omega^{(\pm)} = \mathbf{1}$, follows

$$(\Omega^{(\pm)})^\dagger H \Omega^{(\pm)} = H_0 \ . \tag{12.98}$$

The last equation leads to the interpretation that $\Omega^{(\pm)}$ can be considered as an operator which transforms H into H_0. From this point of view, we have to conclude that $\Omega^{(\pm)}$ **cannot** be unitary. If it would be, then H and H_0 would have to have the same spectrum.

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Since H_0 has only a continuous spectrum, H could not have bound states, which is clearly not the case. We can say that $\Omega^{(\pm)}$ is then and only then unitary, if H does not have bound states.

Let us consider

$$
\begin{aligned}
SH_0 \;=\; \Omega^{(-)\dagger}\,\Omega^{(+)}\,H_0 \;&=\; \Omega^{(-)\dagger}\,H\,\Omega^{(+)} \\
&=\; H^0\,\Omega^{(-)\dagger}\,\Omega^{(+)} \;=\; H_0 S \; ,
\end{aligned}
\tag{12.99}
$$

which shows that $[S, H_0] = 0$. We also have

$$
\begin{aligned}
H_0 \;|\,\vec{p}\,\rangle \;&=\; E_p \;|\,\vec{p}\,\rangle \\
SH_0 \;|\,\vec{p}\,\rangle \;&=\; E_s\,S\,|\,\vec{p}\,\rangle \;=\; H_0 S\;|\,\vec{p}\rangle \; ,
\end{aligned}
\tag{12.100}
$$

which means that $|\,\vec{p}\,\rangle$ as well as $S\,|\,\vec{p}\,\rangle$ are eigenstates of H_0 with the same eigenvalue E_p. For the expectation values for H_0 in the initial state (before the scattering) and the final state (after the scattering), we find

$$
\begin{aligned}
\langle \psi_b^{(-)} \,|\, H_0 \,|\, \psi_b^{(-)} \rangle \;&=\; \langle S\psi_a^{(+)} \,|\, H_0 \,|\, S\psi_a^{(+)} \rangle \\
&=\; \langle \psi_a^{(+)} \,|\, S^\dagger H_0 S \,|\, \psi_a^{(+)} \rangle \\
&=\; \langle \psi_a^{(+)} \,|\, S^\dagger S\, H_0 \,|\, \psi_a^{(+)} \rangle \\
&=\; \langle \psi_a^{(+)} \,|\, H_0 \,|\, \psi_a^{(+)} \rangle
\end{aligned}
\tag{12.101}
$$

H_0 has a complete spectrum of non-normalizable eigenstates $\{|\,\vec{p}\,\rangle\}$, thus S can be represented in these states. From $[S, H_0] = 0$ follows

$$
\begin{aligned}
0 \;=\; \langle \vec{p}\,' \,|\, [H_0, S] \,|\, \vec{p}\,\rangle \;&=\; \langle \vec{p}\,' \,|\, H_0 S \,|\, \vec{p}\,\rangle \;-\; \langle \vec{p}\,' \,|\, SH_0 \,|\, \vec{p}\,\rangle \\
&=\; (E_{p'} - E_p)\langle \vec{p}\,' \,|\, S \,|\, \vec{p}\,\rangle \; .
\end{aligned}
\tag{12.102}
$$

From this follows that $\langle \vec{p}\,' \,|\, S \,|\, \vec{p}\,\rangle \neq 0$ only if $E_{p'} = E_p$. Therefore, we can write

$$
S_{p'p} \;=\; \langle \vec{p}\,' \,|\, S \,|\, \vec{p}\,\rangle \;=\; \delta(E_{p'} - E_p)\langle \hat{p}' \,|\, S \,|\, \hat{p}\rangle
\tag{12.103}
$$

where $\hat{p} = \vec{p}/\,|\,\vec{p}\,|$. This means $S_{p'p}$ is defined on-shell.

12.12   Unitarity of the S-Operator

In order to show that S as defined in (12.97) is unitary, we have to show that

$$S^\dagger S \;=\; SS^\dagger \;=\; \mathbf{1} \;. \qquad (12.104)$$

The first relation is easy and we use that $\Omega^{(+)\dagger} \, \Omega^{(\pm)} = \mathbf{1}$:

$$
\begin{aligned}
\langle \Omega^{(\pm)}\varphi \mid \Omega^{(\pm)}\varphi \rangle \;&=\; \lim_{t'\to\mp\infty} \lim_{t\to\mp\infty} \langle e^{iHt'}\, e^{-iH_0 t'}\, \varphi \mid e^{iHt}\, e^{-iH_0 t}\, \varphi \rangle \\
&\overset{t'=t}{=}\; \lim_{t\to\mp\infty} \langle e^{iHt}\, e^{-iH_0 t}\, \varphi \mid e^{iHt}\, e^{iH_0 t}\, \varphi \rangle \\
&=\; \langle \varphi \mid \mathbf{1} \mid \varphi \rangle \;=\; \| \, \varphi \, \|^2 \;.
\end{aligned}
$$

$$(12.105)$$

In order to show the other relation, we need to consider the mapping properties of $\Omega^{(\pm)\dagger}$. Applied on a scattering state, it gives

$$\Omega^{(\pm)\dagger} \mid \psi_a^{(\pm)} \rangle \;=\; \Omega^{(\pm)\dagger}\, \Omega^{(\pm)} \mid \varphi_a \rangle \;=\; \mid \varphi_a \rangle \;. \qquad (12.106)$$

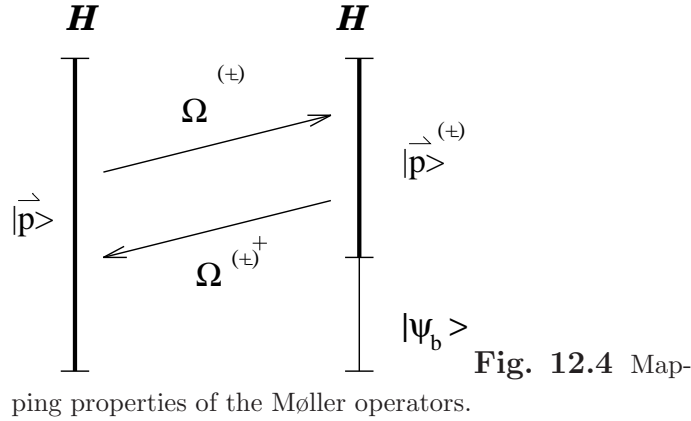This means, applied on a scattering state $\Omega^{(\pm)\dagger}$, gives a full wave packet.

Fig. 12.4 Mapping properties of the Møller operators.

Fig. 12.4 illustrates that

$$
\begin{aligned}
\mid p \rangle^{(\pm)} \;&=\; \Omega^{(\pm)} \mid \vec{p} \, \rangle \\
\mid \vec{p} \, \rangle \;&=\; \Omega^{(\pm)\dagger} \mid \vec{p} \, \rangle^{(\pm)} \quad \text{with } \; Ker(\Omega^{(\pm)\dagger}) \;\neq\; 0 \\
0 \;&=\; \Omega^{(\pm)\dagger} \mid \psi_n \rangle \;.
\end{aligned}
$$

$$(12.107)$$

Thus $\Omega^{(\pm)}\,\Omega^{(\pm)\dagger} = P_c$, a projection operator on the continuous spectrum. Now consider

$$
\begin{aligned}
S^\dagger S &= (\Omega^{(-)\dagger}\Omega^{(+)})^\dagger(\Omega^{(-)\dagger}\Omega^{(+)}) &= \Omega^{(+)\dagger}\Omega^{(-)}\Omega^{(-)\dagger}\,\Omega^{(+)} \\
&&= \Omega^{(+)\dagger}\,\mathbf{P}_c\Omega^{(+)} = \mathbf{1} \\
SS^\dagger &= (\Omega^{(-)\dagger}\Omega^{(+)})(\Omega^{(-)\dagger}\Omega^{(+)})^\dagger &= \Omega^{(-)\dagger}\Omega^{(+)}\Omega^{(+)\dagger}\Omega^{(-)} \\
&&= \Omega^{(-)\dagger}\mathbf{P}_c\,\Omega^{(-)} = \mathbf{1}\ .
\end{aligned}
$$

$$(12.108)$$

As a remark: S is only unitary on the whole Hilbertspace $\mathcal{H}$ if the states $\mid p\rangle^{(+)}$ span the entire space of scattering states.

Let us now consider the probability of scattering into all possible final states

$$
\begin{aligned}
\int d^3p \ \mid \langle \vec{p}\mid S \mid \varphi_a\rangle \mid^2 &= \int d^3p \ \langle \vec{p}\mid S \mid \varphi_a\rangle^* \langle \vec{p}\mid S \mid \varphi_a\rangle \\
&= \int d^3p \ \langle \varphi_a \mid S^\dagger \mid \vec{p}\,\rangle\langle \vec{p}\, S \mid \varphi_a\rangle \\
&= \langle \varphi_a \mid S^\dagger S \mid \varphi_a\rangle = \parallel \varphi_a \parallel^2 \ .
\end{aligned}
$$

$$(12.109)$$

Thus, the probability is conserved in the scattering process.

12.13   Transition Operator

In (12.96) the scattering operator was defined as the transition probability

$$
\begin{aligned}
S_{ba} &= \langle \psi_b^{(-)} \mid \psi_a^{(+)}\rangle \\
&= \langle \varphi_b \mid \psi_a^{(+)}\rangle + \langle \frac{1}{E_a - i\varepsilon - H}\, V\, \varphi_b \mid \psi_a^{(+)}\rangle \ ,
\end{aligned}
$$

$$(12.110)$$

where we used the Low equation (12.79) for the last step. One also has

$$
\begin{aligned}
\langle \varphi_b \mid \psi_a^{(+)}\rangle &= \langle \varphi_b \mid \varphi_a\rangle + \langle \varphi_b \mid G_0(E_a + i\varepsilon)\, V\ \mid \psi_a^{(\pm)}\rangle\ . \\
&= \delta_{ba} + \frac{1}{E_a - E_b - i\varepsilon}\, \langle \varphi_b \mid V \mid \psi_a^{(+)}\rangle
\end{aligned}
$$

$$(12.111)$$

Inserting (12.111) into (12.110) leads to

$$
\begin{aligned}
S_{ba} &= \delta_{ba} + \left[\frac{1}{E_b - E_a + i\varepsilon} - \frac{1}{E_b - E_a - i\varepsilon} \right] \langle \varphi_b \mid V \mid \psi_a^{(+)} \rangle \\
&= \delta_{ba} - 2\pi i \, \delta(E_b - E_a) \, \langle \varphi_b \mid T^{(+)}(E_a) \mid \varphi_a \rangle \\
&= \delta_{ba} - 2\pi i \, \delta(E_b - E_a) \, T_{ab}^{(+)}(E_a) \; .
\end{aligned}
\tag{12.112}
$$

This suggests that the S operator can be split into two parts according to

$$
S = \mathbf{1} - 2\pi i \, \tau^{(+)}
\tag{12.113}
$$

where

$$
\tau_{ba}^{(\pm)} := \delta(E_b - E_a) \, T_{ba}^{(\pm)}
\tag{12.114}
$$

describes the scattering. For the interaction $V = 0$ follows $S = \mathbf{1}$.

Thus, $\tau_{ba}^{\pm}$ describes the scattering process and has to appear in the differential cross section. Thus for scattering relevant quantity is $S - \mathbf{1}$, where the scattering in forward direction is subtracted. Due to the δ function $\delta(E_b - E_a)$, $\tau_{ba}^{(\pm)}$ is only defined for $E_b = E_a$, i.e., on-shell. Another way to formulate this is that the S operator only defines the on-shell elements of the T operator. However, this is not enough to define an operator – one has to know the matrix elements for all momenta. From the unitarity of the S operator follows

$$
\begin{aligned}
SS^\dagger &= (\mathbf{1} - 2\pi i \tau^{(+)})(\mathbf{1} + 2\pi i \tau^{(-)}) \\
&= \mathbf{1} - 2\pi i(\tau^{(+)} - \tau^{(-)}) + 4\pi^2 \, \tau^{(+)}\tau^{(-)}
\end{aligned}
\tag{12.115}
$$

and from there

$$
\begin{aligned}
\tau^{(+)} - \tau^{(-)} &= -2\pi i \, \tau^{(+)}\tau^{(-)} \\
\delta(E_b - E_a)(T_{ba}^{(+)} - T_{ba}^{(-)}) &= -2\pi i \sum_\alpha \delta(E_b - E_a) \, \delta(E_a - E_\alpha) \, T_{b\alpha}^{(+)}T_{\alpha a}^{(-)}
\end{aligned}
\tag{12.116}
$$

and for $E_0 = E_a$

$$
T_{ba}^{(+)} - T_{ba}^{(-)} = -2\pi i \sum_\alpha \delta(E_b - E_\alpha) \, T_{b\alpha}^{(+)}T_{\alpha a}^{(-)}
\tag{12.117}
$$

or explicitly

$$\langle \vec{p} \mid T^+(E_p) - T^{(-)}(E_p) \mid \vec{p}\,\rangle = -2\pi i \int d^3k \, \delta(E_p - E_\alpha)$$
$$\langle \vec{p} \mid T^{(+)}(E_p) \mid \vec{k}\,\rangle \langle \vec{k} \mid T^{(-)}(E_p) \mid \vec{p}\,\rangle$$
(12.118)

which is the on-shell unitarity relation already derived in (12.89). However, (12.118) contains less information than the unitarity relations derived in Section 12.9, which are given fully-off-shell.

12.14 Cross Section

The cross section should describe the probability that under given initial conditions something is scattered into the direction of $\vec{p}\,'$, where the forward direction should be subtracted. This probability is obviously given by $S - \mathbf{1}$ and can be defined as

$$\begin{aligned}
dW &= d^3p' \mid -2\pi i \langle \vec{p}\,' \mid T \mid \varphi_{\vec{p}}\,\rangle \mid^2 \\
&= p'^2 \, dp' \, d\Omega_{\vec{p}\,'} \mid -2\pi i \langle \vec{p}\,' \mid T \mid \varphi_{\vec{p}}\,\rangle \mid^2 \\
&= m \sqrt{2mE'} \, dE' \, d\Omega_{\vec{p}} \, 4\pi^2 \mid \langle \vec{p}\,' \mid T \mid \varphi_{\vec{p}}\,\rangle \mid^2
\end{aligned}$$
(12.119)

where $p^2 = 2mE$ has been used. Next we need to consider

$$\begin{aligned}
\mid \langle \vec{p}\,' \mid T(E_p) \mid \varphi_{\vec{p}}\,\rangle \mid^2 &= \left| \int d^3p \, \langle \vec{p}\,' \mid T(E_p) \mid \vec{p}\,\rangle \, \tilde{\varphi}_{p_0}(\vec{p}\,) \right|^2 \\
&= \left| \int d^3p \, \delta(E' - E_p) \langle \hat{p}' \mid T(E_p) \mid \hat{p}\rangle \, \tilde{\varphi}_{p_0}(\vec{p}\,) \right|^2 .
\end{aligned}$$
(12.120)

For $\tilde{\varphi}_{p_0}(\vec{p}\,)$ being a wave packet which is sharply peaked at $\vec{p} = \vec{p}_0$ and $\langle \hat{p}' \mid T(E_p) \mid \hat{p}\rangle$ being a slowly varying function, we can write

$$\mid \langle \vec{p}\,' \mid T(E_p) \mid \varphi_{\vec{p}}\,\rangle \mid^2 \simeq \mid \langle \hat{p}' \mid T(E_{p_0}) \mid \hat{p}\rangle \mid^2 \int d^3p \, \delta(E' - E_p) \, \tilde{\varphi}_{p_0}(\vec{p}\,) \mid^2$$

and thus

$$\begin{aligned}
dW &\approx 4\pi^2 \, mp_0 \, d\Omega \, dE \mid \langle \hat{p}' \mid T(E_{p_0}) \mid \hat{p}\rangle \mid^2 \int d^3p \, \delta(E' - E_p) \, \tilde{\varphi}_{p_0}(\vec{p}\,) \mid^2 \\
&= 4\pi^2 \, mp_0 \, d\Omega \, dE \mid \langle \hat{p}' \mid T(E_{p_0}) \mid \hat{p}\rangle \mid^2 \, I(E_p) .
\end{aligned}$$
(12.121)

Needed is the scattering into a definite solid angle $d\Omega$, thus one can integrate over the energy

$$dW(\Omega) = 4\pi^2 \, mp_0 \, d\Omega \, \mid \langle \hat{p}' \mid T(E_{p_0}) \mid \hat{p}_0 \rangle^2 \int dE' \, I(E) \, . \qquad (12.122)$$

The energy δ-function can be written as
$\delta(E' - E) = \frac{m}{p_0}\delta(\mid \vec{p}' \mid - \mid \vec{p}_0 \mid)$ leading to another factor $\frac{m}{p_0}$ in 12.122.

The differential cross section was defined as the flux scattered into a specific solid angle $d\Omega$ divided by the incoming flux:

$$
\begin{aligned}
d\sigma &= \frac{dW(\Omega)}{W_0} \\
&= \frac{(2\pi)^4 \, mp_0 \cdot \frac{m}{p_0} \, \rho_0 \, \Delta t \, \mid \langle \hat{p}' \mid T(E_{p_0}) \mid \hat{p}_a \mid^2 \, d\Omega}{\rho_0 \, \Delta t}
\end{aligned}
$$

$$(12.123)$$

from which follows

$$\frac{d\sigma}{d\Omega} = (2\pi)^4 \, m^2 \, \mid \langle \hat{p}' \mid T(E_{p_0}) \mid \hat{p}_0 \rangle \mid^2 = \mid f(\hat{p}', \hat{p}) \mid^2 \qquad (12.124)$$

which gives the relation between the on-shell t-matrix and the scattering amplitude. For the total cross section this leads to

$$\sigma_{tot} = \int d\Omega \frac{d\sigma}{d\Omega} = (2\pi)^4 m^2 \int d\Omega \mid \langle \hat{p}' \mid T(E_{p_0}) \mid \hat{p} \rangle \mid^2 . \qquad (12.125)$$

12.15 Scattering of Identical Spinless Particles

For identical particles the states have to be symmetric (or antisymmetric) under the exchange of the two particles. Interchanging particle labels corresponds to replacing $\vec{p} = (p, \theta, \phi)$ by $-\vec{p} = (p, \pi - \theta, \phi + \pi)$. Similarly, the scattering amplitude has to be symmetric (antisymmetric) under the exchange of 1 and 2, i.e., we have to consider

$$f(\hat{p}', \hat{p}) \pm f(-\hat{p}', \hat{p}) = f(\theta, \phi) \pm f(\pi - \theta, \phi + \pi) \, , \qquad (12.126)$$

Since for spinless particles, the interaction is rotationally invariant around the axis along the incoming momentum $\vec{p}$, we drop the dependence on ϕ and have

$$
\begin{aligned}
\frac{d\sigma}{d\Omega} &= \mid f_p(\theta) + f_p(\pi - \theta) \mid^2 \\
&= \mid f_p(\theta) \mid^2 + \mid f_p(\pi - \theta) \mid^2 + 2Re(f^*(\theta) \, f(\pi - \theta)) \, .
\end{aligned}
$$

$$(12.127)$$

First, the differential cross section for the scattering of two identical spinless particles is symmetric around 90°. Second, $\frac{d\sigma}{d\Omega}$ contains three terms, the cross sections for the amplitudes for scattering into the direction (θ, ϕ) for particles interference from $z = -\infty$ and $z = +\infty$ and an influence term, which is a typical quantum mechanical effect, since one must add amplitudes and not intensities.

Next, we consider the scattering of two identical spin $\frac{1}{2}$ particles interacting via a spin-independent potential. Since their spin is a good quantum number, we consider $S = 0$ and $S = 1$ states separately. The total wave function must be antisymmetric. For $S = 0, \chi_{s=0}$ is antisymmetric, thus the spatial part must be symmetric. Thus, we obtain for the cross section in the singlet state

$$\frac{d\sigma_s}{d\Omega} = \mid f(\theta) + f(\pi - \theta) \mid^2 \tag{12.128}$$

For $S = 1, \chi_{s=1}$ is symmetric, thus the spatial part must be antisymmetric, and we obtain

$$\frac{d\sigma_t}{d\Omega} = \mid f(\theta) - f(\pi - \theta) \mid^2 . \tag{12.129}$$

Eqs. (12.128) and (12.129) assume that the scattering particles are in a definite spin state, i.e., beam and target are polarized. If this is not the case, i.e., if beam as well as target are unpolarized, i.e., spins are randomly oriented, then we must average over the random spin orientations. This corresponds to the assumption that the four spin states, $S = 0, m_s = 0, S = 1, m_s = \pm 1, 0$ occur with equal frequency. Then the **unpolarized** differential cross section is given by

$$\frac{d\sigma}{d\Omega} (unpol) = \frac{1}{4} \frac{d\sigma_s}{d\Omega} + \frac{3}{4} \frac{d\sigma_t}{d\Omega} . \tag{12.130}$$

12.16   The Gellman-Goldberger Relation

The Gellman-Goldberger relation or **two-potential formula** is particularly appropriate when the interaction between the projectile and the target decomposes naturally into two parts: $V = V_0 + V_1$. This division is especially useful if the scattering wave function under the action of one part can be obtained exactly, while the effect of the other can be treated in the same approximation. In this sense, the formula leads to the so-called "distorted-wave Born approximation" and, in other circumstances, to the method of the "final state interaction."

Let us assume that the interaction potential $V = V_0 + V_1$. Then, the corresponding LS equation reads

$$\begin{aligned} T &= (V_0 + V_1) + (V_0 + V_1) G_0 T \\ &= (V_0 + V_1) + V_0 G_0 T + V_1 G_0 T \end{aligned} \tag{12.131}$$

As a reminder, from (12.58) we had for the scattering state

$$| \vec{p} \, \rangle^{(+)} \; = \; | \vec{p} \, \rangle \; + \; G_0 V \; | \vec{p} \, \rangle^{(+)}$$

from which follows

$$| \vec{p} \, \rangle^{(+)} \; = \; (1 - G_0 V)^{-1} \; | \vec{p} \, \rangle \; = \; \Omega^{(+)} \; | \vec{p} \, \rangle \, . \tag{12.132}$$

From the relation (12.39), we obtained the Low equation in the form

$$| \vec{p} \, \rangle^{(+)} \; = \; | \vec{p} \, \rangle \; + \; G V \; | \vec{p} \, \rangle \; = \; (1 + G V) \; | \vec{p} \, \rangle \; = \; \Omega^{(+)} \; | p \rangle \tag{12.133}$$

So we have the different representations of the Møller operator

$$\Omega^{(+)} \; = \; (1 - G_0 V)^{-1} \; = \; (1 + G V) \; = \; 1 \; + \; G_0 T \, . \tag{12.134}$$

Multiplying (12.131) from the left with $\Omega_0^{(+)\dagger} := (1 - V_0 G_0)^{-1}$ gives

$$(1 - V_0 G_0)^{-1} \, (1 - V_0 G_0) T \; = \; (1 - V_0 G_0)^{-1} \, V_0 + (1 - V_0 G_0)^{-1} \, V_1 \tag{12.135}$$
$$+ \; (1 - V_0 G_0)^{-1} \, V_1 G_0 T$$
$$T \; = \; T_0 \; + \; (1 - V_0 G_0)^{-1} \, V_1 \, (1 + G_0 T)$$
$$\tag{12.136}$$

where we used that $T_0 = V_0 + V_0 \, G_0 T_0$ as exact solution of the Hamiltonian $H_0 + V_0$. Using the relations (12.134), one obtains

$$T \; = \; T_0 \; + \; \Omega_0^{(-)\dagger} \, V_1 \, \Omega^{(+)} \tag{12.137}$$

where $\Omega^{(+)}$ is the Møller operator for the full scattering problem. Eq (12.137) is the so-called "two-potential formula." The first term is the scattering amplitude for the potential V_0 alone; in many cases, it is either known or calculable. The second term incorporates the effect of the "residual potential "V_1 and must often be treated approximately. If the interaction V_1 is sufficiently weak, it can be treated in a first order approximation, which means here $\Omega^{(+)} \approx \Omega_0^{(+)}$ so that

$$T \; \approx \; T_0 \; + \; \Omega_0^{(-)\dagger} \, V_1 \, \Omega_0^{(+)} \tag{12.138}$$

or in terms of matrix elements

$$\langle \vec{p}\,' \,|\, T \,|\, \vec{p}\,\rangle \;:=\; T_{fi} \;=\; \langle \vec{p}\,' \,|\, T_0 \,|\, \vec{p}\,\rangle \;+\; {}^{(-)}\langle \vec{p_0}\,' \,|\, V_1 \,|\, \vec{p_o}\rangle^{(+)} \;. \qquad (12.139)$$

The second term is a generalization of the Born approximation in which distorted wave functions $|\, p_0\rangle^{(+)}$ are used to calculate the scattering due to V_1 in the presence of V_0. It is often called the **distorted wave Born approximation**. For instance, in the scattering of protons from nuclei, one might choose V_0 to be the Coulomb potential and V_1 the short-range nuclear force. Then the second term in (12.139) approximates the scattering due to the nuclear force, taking account of the Coulomb repulsion and the consequent reduction in the amplitude of the wave function at small separations. See Ch. Elster, L. C. Liu and R. M. Thaler, "A Practical calculational method for treating Coulomb scattering in momentum space," J. Phys. G **19**, 2123 (1993).